HOMOLOGICAL ALGEBRA AND SHEAF THEORY

MASTER 2 LECTURE – NANTES – JANUARY 2024

STÉPHANE GUILLERMOU

1. Some definitions about categories and functors

1.1. Categories.

Definition 1.1. A category $\mathcal{C}$ consists of the data of

(i) a set $\mathrm{Ob}(\mathcal{C})$ (the set of *objects*),
(ii) for any $X, Y \in \mathrm{Ob}(\mathcal{C})$, a set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ (the set of *morphisms*),
(iii) for any $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a map (the *composition*)

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Z), \quad (f,g) \mapsto g \circ f,$$

satisfying

(i) $\circ$ is associative, that is, $(h \circ g) \circ f = h \circ (g \circ f)$ as soon as both sides make sense,
(ii) for each $X \in \mathrm{Ob}(\mathcal{C})$, there exists $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$ which is neutral for $\circ$ on the right and on the left, that is, $\mathrm{id}_X \circ f = f$, $g \circ \mathrm{id}_X = g$, as soon as the left hand sides make sense.

We often write $\mathrm{Hom}(X,Y)$ instead of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $X \in \mathcal{C}$ instead of $X \in \mathrm{Ob}(\mathcal{C})$. We also write $f\colon X \to Y$ instead of $f \in \mathrm{Hom}(X,Y)$.

A morphism $f\colon X \to Y$ is an *isomorphism* if there exists $g\colon Y \to X$ (the *inverse* of f) such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. If such a g exists, it is unique.

To avoid logical contradiction we cannot consider the set of all sets. So, when we consider the category **Set** of sets (or the category of groups, rings,…) we assume that we have chosen a set $\mathcal{U}$, called a *universe* which is stable by the operations of set's theory (union, intersection, product,…) and we only consider the categories whose objects and morphisms sets belong to $\mathcal{U}$. For any given set X, there exists a universe containing X. For more details see [3, §I.6].

Examples of categories abound (the category of sets, topological spaces, manifolds, rings,…) but we will soon restrict to categories which are similar in some sense to the category of modules over a ring, so called *abelian* categories.

1.2. **Functors.**

Definition 1.2. Let $\mathcal{C}, \mathcal{C}'$ be two categories. A functor F from $\mathcal{C}$ to $\mathcal{C}'$ is the data of maps (also denoted by F) $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}')$ and $F\colon \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}'}(F(X), F(Y))$, for all $X, Y \in \mathrm{Ob}(\mathcal{C})$, satisfying

(i) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all $X \in \mathrm{Ob}(\mathcal{C})$,
(ii) $F(f \circ g) = F(f) \circ F(g)$, for all composable morphisms f, g.

For two functors $F\colon \mathcal{C} \to \mathcal{C}'$ and $G\colon \mathcal{C}' \to \mathcal{C}''$ we define the composition $G \circ F$ by $(G \circ F)(X) = G(F(X))$ for $X \in \mathrm{Ob}(\mathcal{C})$ and $(G \circ F)(f) = G(F(f))$ for all morphisms f in $\mathcal{C}$.

For a category $\mathcal{C}$ we define the *opposite* category $\mathcal{C}^{\mathrm{op}}$ by $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,Y) := \mathrm{Hom}_{\mathcal{C}}(Y,X)$ for all $X, Y \in \mathrm{Ob}(\mathcal{C})$.

A *contravariant* functor from $\mathcal{C}$ to $\mathcal{C}'$ is a functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{C}'$. (Functors can be called covariant functors if we want to insist.)

Example 1.3. For a category $\mathcal{C}$ and $X \in \mathcal{C}$ we define a functor $h(X)\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, $Y \mapsto \mathrm{Hom}_{\mathcal{C}}(Y,X)$.

Definition 1.4. Let $\mathcal{C}, \mathcal{C}'$ be two categories and let F, G be two functors from $\mathcal{C}$ to $\mathcal{C}'$. A morphism of functors θ from F to G is the data of morphisms $\theta_X\colon F(X) \to G(X)$ for all $X \in \mathrm{Ob}(\mathcal{C})$ such that, for all morphisms $f\colon X \to Y$ in $\mathcal{C}$, the following diagram commutes

$$\begin{array}{ccc} F(X) & \xrightarrow{\theta_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[\theta_Y]{} & G(Y). \end{array}$$

In this way the set $\mathrm{Fct}(\mathcal{C}, \mathcal{C}')$ of functors from $\mathcal{C}$ to $\mathcal{C}'$ becomes a category.

Example 1.5. We set $\mathcal{C}^\wedge = \mathrm{Fct}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$. The above functors $h(X)$, $X \in \mathcal{C}$, define a functor $h\colon \mathcal{C} \to \mathcal{C}^\wedge$, called the Yoneda functor.

Definition 1.6. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a functor. We say that F is full (resp. faithful, fully faithful) if the maps $F\colon \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}'}(F(X), F(Y))$ are surjective (resp. injective, bijective), for all $X, Y \in \mathrm{Ob}(\mathcal{C})$.

We say that F is essentially surjective if for each $Y \in \mathrm{Ob}(\mathcal{C}')$ there exist $X \in \mathrm{Ob}(\mathcal{C})$ and an isomorphism $F(X) \simeq Y$.

We say that F is an equivalence of categories if there exist a functor $G\colon \mathcal{C}' \to \mathcal{C}$ and isomorphisms of functors $\mathrm{id}_{\mathcal{C}} \simeq G \circ F$ and $\mathrm{id}_{\mathcal{C}'} \simeq F \circ G$. We then write $F\colon \mathcal{C} \xrightarrow{\sim} \mathcal{C}'$ and we say that F and G are quasi-inverse to each other.

For example the category of finite dimensional vector spaces over some field $\mathbf{k}$, say $\mathbf{Vect}_f(\mathbf{k})$, is equivalent to its *full* subcategory $\mathbf{Mat}(\mathbf{k})$ with $\mathrm{Ob}(\mathbf{Mat}(\mathbf{k})) = \{\mathbf{k}^n; n \in \mathbb{N}\}$ (where full means that the Hom sets are the same: $\mathrm{Hom}_{\mathbf{Mat}(\mathbf{k})}(\mathbf{k}^n, \mathbf{k}^m) = \mathrm{Hom}_{\mathbf{Vect}_f(\mathbf{k})}(\mathbf{k}^n, \mathbf{k}^m) = \mathrm{Mat}(m \times n, \mathbf{k})$).

Exercise 1.7. A functor $F\colon \mathcal{C} \to \mathcal{C}'$ is an equivalence of categories if and only if it is fully faithful and essentially surjective.

Exercise 1.8. For any category $\mathcal{C}$ the Yoneda functor

$$h\colon \mathcal{C} \to \mathcal{C}^\wedge = \mathrm{Fct}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$$

is fully faithful.

2. Example of derived functors: extension

Let R be a ring and let $\mathrm{Mod}(R)$ be the category of left R-modules and R-linear maps. In the case $R = \mathbb{Z}$ we remark that $\mathrm{Mod}(\mathbb{Z}) = \mathbf{Ab}$, the category of abelian groups and additive morphisms.

2.1. **Exact sequences, exact functors.** A composable pair of morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathrm{Mod}(R)$ is "exact at B" if $\ker g = \mathrm{im}\, f$. A long exact sequence is a sequence $\cdots \to A^n \xrightarrow{d^n} A^{n+1} \to \cdots$, $n \in \mathbb{Z}$, which is exact at each A^n, $n \in \mathbb{Z}$. A short exact sequence is a sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ which is exact at A, B and C, that is, f is injective and $C \simeq B/A$.

A functor $F\colon \mathrm{Mod}(R) \to \mathbf{Ab}$ is *additive* if the maps $\mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$, $f \mapsto F(f)$, are group morphisms, for all $A, B \in \mathrm{Mod}(R)$.

An additive functor $F\colon \mathrm{Mod}(R) \to \mathbf{Ab}$ is *exact* if it sends short exact sequences to short exact sequences. It is *left exact* if, for any exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C$, the sequence $0 \to F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$ is exact. It is *right exact* if, for any exact sequence $A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the sequence $F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \to 0$ is exact.

Example 2.1. For any $M \in \mathrm{Mod}(R)$, both functors

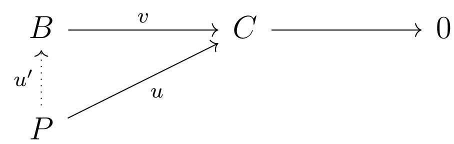
$$\begin{aligned} \mathrm{Hom}(M, -)\colon \mathrm{Mod}(R) &\to \mathbf{Ab}, & X &\mapsto \mathrm{Hom}(M, X) \\ \mathrm{Hom}(-, M)\colon \mathrm{Mod}(R)^{\mathrm{op}} &\to \mathbf{Ab}, & X &\mapsto \mathrm{Hom}(X, M) \end{aligned}$$

are left exact.

2.2. **Projectives, injectives, derived functors.** The starting point of homological algebra is that it makes sense, for a given left (or right) exact functor, to "measure" its deviation from being exact. For example there exists a first *derived* functor of the functor $\mathrm{Hom}(-, M)$, denoted $\mathrm{Ext}^1(-, M)$ such that, for any exact sequence $0 \to A \xrightarrow{u} B \xrightarrow{v} C \to 0$, there exists an exact sequence $0 \to \mathrm{Hom}(C, M) \to \mathrm{Hom}(B, M) \to \mathrm{Hom}(A, M) \to \mathrm{Ext}^1(C, M)$. The fact that $\mathrm{Hom}(-, M)$ is not right exact means that $\mathrm{Ext}^1(-, M)$ is not the zero functor.

Definition 2.2. An R-module P is projective if, for any given surjective morphism $v\colon B \to C$ in $\mathrm{Mod}(R)$ and any $u\colon P \to C$, there exists

$u' \colon P \to B$ such that $u = v \circ u'$:



We can rephrase the definition by saying that P is projective if, for any short exact sequence $B \to C \to 0$, the sequence $\mathrm{Hom}(P, B) \to \mathrm{Hom}(P, C) \to 0$ is exact (it then follows that the functor $\mathrm{Hom}(P, \cdot)$ is exact). Projective modules have a good behaviour with respect to $\mathrm{Hom}(-, M)$:

Lemma 2.3. *Let* $0 \to A \xrightarrow{u} B \xrightarrow{v} P \to 0$ *be an exact sequence in* $\mathrm{Mod}(R)$*. We assume that* P *is projective. Then the sequence* $0 \to \mathrm{Hom}(P, M) \to \mathrm{Hom}(B, M) \to \mathrm{Hom}(A, M) \to 0$ *is exact.*

The idea is to replace an arbitrary R-module by a sequence of projectives.

Definition 2.4. Let $A \in \mathrm{Mod}(R)$. A left resolution of A is a long exact sequence

$$\cdots \to P^i \xrightarrow{d^i} P^{i+1} \to \cdots \to P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{\varepsilon} A \to 0.$$

More precisely, the resolution is $(P^{\cdot}, d^{\cdot})$ and ε is the *augmentation* morphism. It is called a projective resolution if all the P^i's are projective modules.

Proposition 2.5. *Let* $A \in \mathrm{Mod}(R)$*. Let* $(P^{\cdot}, d^{\cdot})$ *be a projective left resolution of* A *and let*

$$0 \to \mathrm{Hom}(P^0, M) \xrightarrow{e^0} \mathrm{Hom}(P^1, M) \to \cdots$$
$$\to \mathrm{Hom}(P^{i-1}, M) \xrightarrow{e^{i-1}} \mathrm{Hom}(P^i, M) \to \cdots$$

be the sequence obtained by applying $\mathrm{Hom}(-, M)$ *to this resolution, where* $e^i = \mathrm{Hom}(d^{-i-1}, M)$*. Then* $\mathrm{Ext}^i(A, M) := \ker(e^i)/\operatorname{im}(e^{i-1})$ *is independent of the choice of* $\{P^{\cdot}, d^{\cdot}\}$*.*

Then $A \mapsto \mathrm{Ext}^i(A, M)$ is a functor, called the i^{th} *derived functor* of $\mathrm{Hom}(-, M)$ (also called i^{th} extension group in this case). We can check that $\mathrm{Ext}^0(A, M) = \mathrm{Hom}(A, M)$.

Proposition 2.6. *Let* $0 \to A \xrightarrow{u} B \xrightarrow{v} C \to 0$ *be an exact sequence in* $\mathrm{Mod}(R)$*. Then there exists a long exact sequence*

$$\begin{aligned} 0 \to \mathrm{Hom}(C, M) \to \mathrm{Hom}(B, M) \to \mathrm{Hom}(A, M) \to \mathrm{Ext}^1(C, M) \\ \to \cdots \to \mathrm{Ext}^i(C, M) \to \mathrm{Ext}^i(B, M) \to \mathrm{Ext}^i(A, M) \\ \xrightarrow{\delta^i} \mathrm{Ext}^{i+1}(C, M) \to \mathrm{Ext}^{i+1}(B, M) \to \cdots \end{aligned}$$

Reversing the arrows we will also define the notion of *injective* objects and use them to define the *derived functors* of $\mathrm{Hom}(M, -)$ (it turns out that they are the same: $\mathrm{Ext}^i(M, A)$ but this is not a priori obvious).

We will give a proof in a more general framework, that of abelian categories. Our main example will be the category of sheaves on a topological space X. Then the derived functors of the "global section" functor will recover the cohomology groups of X, which are the first invariants associated with a manifold.

3. Sheaves

Definition 3.1. Let X be a topological space. A *presheaf* P of abelian groups on X is the data of

(i) an abelian group $P(U)$, for each open subset $U \subset X$, the group of *sections over* U,
(ii) a morphism of groups $r_V^U \colon P(U) \to P(V)$, for each inclusion of open subsets $V \subset U \subset X$, the *restriction map*, also denoted $s \mapsto s|_V$,

satisfying

(i) $r_U^U = \mathrm{id}_{P(U)}$, for each open subset $U \subset X$,
(ii) $r_W^V \circ r_V^U = r_W^U$, for each inclusion of three open subsets $W \subset V \subset U \subset X$.

A morphism of presheaves $f \colon P \to P'$ is the data of groups morphisms $f(U) \colon P(U) \to P'(U)$ which commute with the restriction maps, that is, $r'_{V,U} \circ f(U) = f(V) \circ r_V^U$, for all $V \subset U \subset X$.

Remark 3.2. Let $\mathrm{Op}(X)$ be the category with objects the open subsets of X and morphisms the inclusions, that is, $\mathrm{Hom}_{\mathrm{Op}(X)}(U, V)$ is a set with one object if $U \subset V$ and is empty if $U \not\subset V$. There is only one possibility for the composition law. Then a presheaf on X is a contravariant functor from $\mathrm{Op}(X)$ to **Ab**.

More generally, for a ring R we can define presheaves of R-modules by replacing abelian groups by R-modules in the definition. A presheaves of R-modules on X is a contravariant functor from $\mathrm{Op}(X)$ to $\mathrm{Mod}(R)$.

Examples 3.3. 1) Let M be an abelian group. The *constant presheaf* of group M on X is the presheaf PM_X defined by $PM_X(U) = M$ for all open subsets $U \subset X$ and $r_V^U = \mathrm{id}_M$ for $V \subset U$.

2) We let $\mathcal{C}_X^0(U)$ be the space of continuous functions (with values in $\mathbb{C}$) on an open subset $U \subset X$. Then $U \mapsto \mathcal{C}_X^0(U)$ and the obvious restriction maps define a presheaf $\mathcal{C}_M^0$. When X is a $\mathcal{C}^\infty$-manifold we define in the same way the presheaf of $\mathcal{C}^\infty$-functions, denoted $\mathcal{C}_X^\infty$.

3) Let X be a topological space endowed with a measure μ. We let L_X^1 be the presheaf of integrable functions defined by $L_X^1(U) = \{f \colon U \to \mathbb{C};$ f is measurable and $\int_U |f| d\mu < \infty\}$.

Definition 3.4. Let X be a topological space. A *sheaf* F of abelian groups on X is a presheaf satisfying

(i) **separation:** for any open subset $U \subset X$, any open covering $U = \bigcup_{i \in I} U_i$ and any section $s \in F(U)$, if $s|_{U_i} = 0$ for all $i \in I$, then $s = 0$,

(ii) **gluing:** for any open subset $U \subset X$, any open covering $U = \bigcup_{i\in I} U_i$ and any collection of sections $s_i \in F(U_i)$, which are compatible in the sense that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all $i,j \in I$, there exists a section $s \in F(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

A morphism of sheaves $f\colon F \to F'$ is a morphism of the underlying presheaves. We denote by $\mathbf{Psh}(X)$ (resp. $\mathbf{Sh}(X)$) the category of presheaves (resp. sheaves) on X.

Remark 3.5. In Definition 3.1 it is allowed to take the empty family I as the set of indices for a covering. It turns out that it makes sense to ask what is the union of an arbitrary family $\bigcup_{i\in I} X_i$ (below we recall the definition of a coproduct of two objects; we can extend to an arbitrary family) even when $I = \emptyset$: the result is $\bigcup_{i\in\emptyset} X_i = \emptyset$. (What is the product of an empty family of sets, $\prod_{i\in\emptyset} X_i = ?$)

Now we can apply the separation axiom with the covering $\emptyset = \bigcup_{i\in\emptyset} U_i$ of the empty set. Take $s \in F(\emptyset)$. The condition "$s|_{U_i} = 0$ for all $i \in I$" is automatically satisfied since there is nothing to check. So we obtain $s = 0$.

In conclusion, for any sheaf F we have $F(\emptyset) = 0$.

Examples 3.6. The presheaves $\mathcal{C}^0_X$ and $\mathcal{C}^\infty_X$ are sheaves. The presheaves PA_X and L^1_X are not.

Given a presheaf P there exists a "closest possible" sheaf corresponding to P, called the associated sheaf of P and denoted by P^a. We will see a more precise definition when we introduce adjoint functors. For the moment we give an ad hoc definition of P^a.

Definition 3.7. Let X be a topological space and let $P \in \mathbf{Psh}(X)$. For a given point $x \in X$ we set $P_x = \varinjlim_{x\in U} P(U)$, where U runs over the open neighborhoods of x. In other words $P_x = (\bigsqcup_{x\in U} P(U))/\sim$ where $\sim$ is the equivalence relation defined for $s \in P(U)$, $t \in P(V)$ by $s \sim t$ if there exists a third neighborhood of x, $W \subset U \cap V$, such that $s|_W = t|_W$.

The group P_x is called the stalk of P at x. For $s \in P(U)$ its image in P_x is denoted s_x and called the germ of s at x.

For a morphism $u\colon P \to Q$ in $\mathbf{Psh}(X)$ we denote by $u_x\colon P_x \to Q_x$ the induced morphism on the stalks.

Lemma 3.8. *Let F be a sheaf on X and $s \in F(U)$ for some open subset U. Then $s = 0$ if and only if $s_x = 0$ for all $x \in U$.*

Proposition 3.9. *Let $u\colon F \to G$ be a morphism in $\mathbf{Sh}(X)$. Then u is an isomorphism if and only if u_x is an isomorphism for all $x \in X$.*

Proof. See for example [1] Prop. 2.2.2. □

Proposition 3.10. *Let X be a topological space and let $P \in \mathbf{Psh}(X)$. There exist a sheaf P^a and a morphism of presheaves $u\colon P \to P^a$ such that u_x is an isomorphism, for each $x \in X$. Moreover the pair (P^a, u) is unique up to isomorphism.*

Proof. See for example [1] Prop. 2.2.3. We only give a definition of P^a.

For an open set $U \subset X$ we set $P^a(U) = \{s = (s(x))_{x \in U} \in \prod_{x \in U} P_x;$ for all $x \in U$ there exists a neighborhood V of x in U and $t \in P(V)$ such that $s(y) = t_y$ for all $y \in V\}$. □

Examples 3.11. 1) Let A be an abelian group. The *constant sheaf* of group A on X is the sheaf associated with PA_X, denoted $A_X = (PA_X)^a$. We have $A_X(U) = \{f\colon U \to A;\ f \text{ is locally constant}\}$, where a function f is said locally constant if for any $x \in U$ there exists a neighborhood V of x in U such that $f|_V$ is a constant function. If X is locally connected, we have $A_X(U) \simeq A^{\pi_0(U)}$, where $\pi_0(U)$ is the set of connected components.

2) Let X be a topological space endowed with a measure μ. Then $(L^1_X)^a = L^{1,loc}_X$ where $L^{1,loc}_X(U) = \{f\colon U \to \mathbb{C};\ f$ is measurable and locally integrable$\}$.

4. Some properties of categories

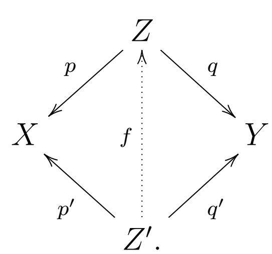
We want to define products, kernels,... in an arbitrary category. The general procedure is to use the Yoneda embedding $h\colon \mathcal{C} \to \mathcal{C}^\wedge = \mathrm{Fct}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$ (or a variation $\mathcal{C} \to \mathrm{Fct}(\mathcal{C}^{\mathrm{op}}, \mathbf{Ab})$ for an additive category). Then we consider the corresponding operation on $\mathrm{Fct}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$, induced from **Set**, and we ask whether the result belongs to the image of the Yoneda embedding. We recall that h is fully faithful and we have the remark:

Exercise 4.1. A fully faithful functor $F\colon \mathcal{C} \to \mathcal{C}'$ is *conservative*: for $u\colon X \to Y$ dans $\mathcal{C}$, if $F(u)$ is an isomorphism, then u is an isomorphism.

In particular $h(Z) \simeq h(Z')$ implies $Z \simeq Z'$. Hence if $F \in \mathcal{C}^\wedge$ is of the form $F \simeq h(Z)$, then Z is well-defined up to isomorphism. We say F is *representable* by Z.

For example, we want to define the product Z of $X, Y \in \mathcal{C}$. We have $h(X), h(Y)$ in $\mathcal{C}^\wedge$. In $\mathcal{C}^\wedge$ we define the product by $(F \times G)(X) := F(X) \times G(X)$ (check that this defines an object of $\mathcal{C}^\wedge$). Then we say that X, Y have a product if there exists $Z \in \mathcal{C}$ such that $h(Z) = h(X) \times h(Y)$. If this is the case, then Z is well-defined up to a unique isomorphism. Here is an equivalent definition:

Definition 4.2. Let $\mathcal{C}$ be a category and $X, Y \in \mathrm{Ob}(\mathcal{C})$. A product of X and Y is an object Z together with morphisms $p\colon Z \to X$, $q\colon Z \to Y$ such that, for any other Z' and $p'\colon Z' \to X$, $q'\colon Z' \to Y$ there exists a unique $f\colon Z' \to Z$ such that $p' = p \circ f$ and $q' = q \circ f$:



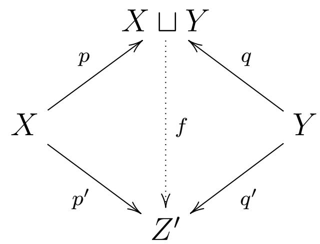
If it exists, the product is unique up to a unique isomorphism. It is denoted $X \times Y$.

We can rephrase the definition of the product by

$$\mathrm{Hom}(Z', X \times Y) = \mathrm{Hom}(Z', X) \times \mathrm{Hom}(Z', Y), \quad \text{for all } Z' \in \mathrm{Ob}(\mathcal{C}),$$

where the second $\times$ is the product in the category of sets.

A coproduct is defined by reversing the arrows. It is often denoted $X \sqcup Y$



We have $\mathrm{Hom}(X \sqcup Y, Z') = \mathrm{Hom}(X, Z') \times \mathrm{Hom}(Y, Z')$.

An object X in a category $\mathcal{C}$ is called *initial* if $\mathrm{Hom}(X, Y)$ consists of a single element for all $Y \in \mathrm{Ob}(\mathcal{C})$. It is called *final* if $\mathrm{Hom}(Y, X)$ consists of a single element for all $Y \in \mathrm{Ob}(\mathcal{C})$. It is called a *zero object* if it is both final and initial. Final, initial or zero objects are unique up to a unique isomorphism, if they exist.

A zero object is usually denoted by 0. If it exists, we also denote by $0 \in \mathrm{Hom}(X, Y)$, for any objects X, Y, the morphism given by the composition $X \to 0 \to Y$. We remark that $0 \circ f = f \circ 0 = 0$ for any f.

Remark 4.3. Let $\mathcal{C}$ be a category, $X, Y \in \mathcal{C}$. Let $h \colon \mathcal{C} \to \mathrm{Fct}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$ be the Yoneda functor of $\mathcal{C}$, and $h' \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Fct}(\mathcal{C}, \mathbf{Set})$ the Yoneda functor of $\mathcal{C}^{\mathrm{op}}$. We could define "opposite" notions, $\widehat{\times}$ and $\widehat{\sqcup}$ by:
(i) $X \widehat{\sqcup} Y$ (if it exists) is characterized by $h(X \widehat{\sqcup} Y) = h(X) \sqcup h(Y)$,
(ii) $X \widehat{\times} Y$ (if it exists) is characterized by $h'(X \widehat{\times} Y) = h'(X) \sqcup h'(Y)$.

However, if $\mathcal{C}$ has a initial (respectively final) object, $\widehat{\sqcup}$ (respectively $\widehat{\times}$) does not exist.

Definition 4.4. A category $\mathcal{C}$ is *additive* if it satisfies the conditions:

(1) for any $X, Y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is an abelian group,
(2) the composition law is bilinear,
(3) there exists a zero object in $\mathcal{C}$,
(4) $\mathcal{C}$ admits products and coproducts.

Note that $\mathrm{Hom}(X, Y) \neq \emptyset$ since it is a group and for all $X \in \mathcal{C}$, $\mathrm{Hom}(X, 0) = \mathrm{Hom}(0, X) = 0$.

In an additive category the coproduct is called the sum and denoted $\oplus$ instead of $\sqcup$.

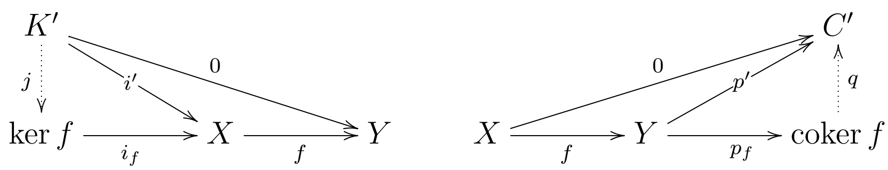
Exercise 4.5. Since $\mathrm{Hom}(X, X \times Y) = \mathrm{Hom}(X, X) \times \mathrm{Hom}(X, Y)$, $(\mathrm{id}_X, 0)$ gives a morphism $i_X \colon X \to X \times Y$. In the same way we define $i_Y \colon Y \to X \times Y$.

We also have $\mathrm{Hom}(X \oplus Y, Z) = \mathrm{Hom}(X, Z) \times \mathrm{Hom}(X, Z)$ and (i_X, i_Y) defines $u \colon X \oplus Y \to X \times Y$. Prove that u is an isomorphism.

Definition 4.6. Let $\mathcal{C}$ be an additive category and let $f\colon X \to Y$ be a morphism in $\mathcal{C}$. A *kernel* of f is a morphism $i_f\colon K \to X$ such that $f \circ i_f = 0$ and such that, for any morphism $i'\colon K' \to X$ satisfying $f \circ i' = 0$ there exists a unique $j\colon K' \to K$ such that $i' = i_f \circ j$. If the kernel exists, it is unique up to a unique isomorphism and we set $\ker f = K$.

A *cokernel* of f is a kernel in the opposite category. It is denoted $\operatorname{coker} f$.

This is visualized by the diagrams:



We can also rephrase the definitions by, for all $Z \in \mathrm{Ob}(\mathcal{C})$:

$$\mathrm{Hom}(Z, \ker(f)) \simeq \ker(\varphi_f\colon \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z, Y)),$$
$$\mathrm{Hom}(\operatorname{coker}(f), Z) \simeq \ker(\psi_f\colon \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)),$$

with $\varphi_f(u) = f \circ u$, $\psi_f(u) = u \circ f$.

Definition 4.7. A morphism $f\colon X \to Y$ in a category $\mathcal{C}$ is called a *monomorphism* if, for all $W \in \mathrm{Ob}(\mathcal{C})$ and all morphisms $g, h\colon W \to X$ in $\mathcal{C}$, the equality $f \circ g = f \circ h$ implies $g = h$ (in other words, the map $\mathrm{Hom}_{\mathcal{C}}(W, X) \to \mathrm{Hom}_{\mathcal{C}}(W, Y)$, $g \mapsto f \circ g$, is injective).

Similarly f is called an *epimorphism* if, for all $g, h\colon Y \to W$, $g \circ f = h \circ f$ implies $g = h$ (or, equivalently, $\mathrm{Hom}_{\mathcal{C}}(Y, W) \to \mathrm{Hom}_{\mathcal{C}}(X, W)$ is surjective).

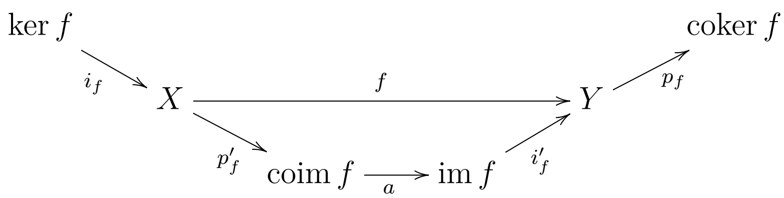
Exercise 4.8. If $\mathcal{C}$ is an additive category, prove that f is an monomorphism if and only if the kernel of f exists and is 0. Prove that f is an epimorphism if and only if $\operatorname{coker} f \simeq 0$.

Exercise 4.9. Let $\mathcal{C}$ be an additive category and let $f\colon X \to Y$ be a morphism in $\mathcal{C}$. We assume that $\ker f$ exists. Prove that the morphism $i_f\colon \ker f \to X$ is a monomorphism.

Dually, if f has a cokernel, $p_f\colon Y \to \operatorname{coker} f$ is an epimorphism.

5. Abelian categories

Lemma 5.1. *Let $\mathcal{C}$ be an additive category and let $f\colon X \to Y$ be a morphism which admits a kernel* $\ker f \xrightarrow{i_f} X$ *and a cokernel* $Y \xrightarrow{p_f} \operatorname{coker} f$*. We also assume that i_f has a cokernel (it is called the* coimage *of f, say* $X \xrightarrow{p'_f} \operatorname{coim} f$*) and that p_f has a kernel (it is called the* image *of f, say* $\operatorname{im} f \xrightarrow{i'_f} Y$*). Then there exists a unique morphism* $a\colon \operatorname{coim} f \to \operatorname{im} f$ *such that $f = i'_f \circ a \circ p'_f$.*



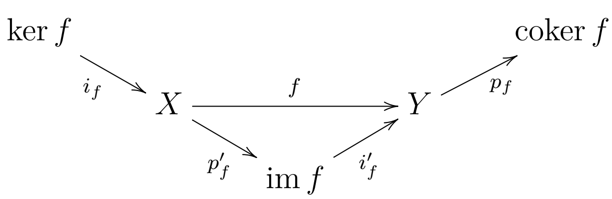
Proof. The existence of a follows from the universal properties of ker and coker. If we have another a' making the diagram commute, then $a\circ p'_f = a' \circ p'_f$ because i'_f is a monomorphism. And then $a = a'$ because p'_f is an epimorphism. ☐

Definition 5.2. An *abelian* category $\mathcal{C}$ is an additive category such that, for any morphism $f\colon X \to Y$, the kernel and the cokernel of f exist (hence also the image and the coimage) and the natural morphism $\operatorname{coim} f \to \operatorname{im} f$ of Lemma 5.1 is an isomorphism.

The typical example is the category **Ab** of abelian groups.

In an abelian category we will identify $\operatorname{coim} f$ and $\operatorname{im} f$: the diagram of the Lemma 5.1 becomes

(5.1)



and, by the result of Exercise 4.9, we have short exact sequences:

$$
\begin{aligned}
&0 \to \ker f \xrightarrow{i_f} X \xrightarrow{p'_f} \operatorname{im} f \to 0, \\
&0 \to \operatorname{im} f \xrightarrow{i'_f} Y \xrightarrow{p_f} \operatorname{coker} f \to 0.
\end{aligned}
\tag{5.2}
$$

Example 5.3. Let $\mathbf{k}$ be a field and $\mathbf{Vect}_{fil}$ be the category of filtered vector spaces over $\mathbf{k}$. The objects, denoted $(V, F^{\cdot})$, are vector spaces V together with sequences of subspaces $\cdots F^iV \subset F^{i+1}V \subset F^{i+2}V \cdots \subset$

V where $i \in \mathbb{Z}$ such that $V = \bigcup_{i\in\mathbb{N}} V_i$. The morphisms from $(V, F^{\cdot})$ to $(W, F^{\cdot})$ are linear maps $u\colon V \to W$ such that $u(F^iV) \subset F^iW$.

The category $\mathbf{Vect}_{fil}$ is an additive category with kernels and cokernels. For $u\colon (V, F^{\cdot}) \to (W, F^{\cdot})$, we can check that $\ker u$ is the usual $\ker u$ with the filtration $F^i(\ker u) = \ker(u|_{F^iV})$ and $\operatorname{coker} u$ is the usual $\operatorname{coker} u$ with the filtration $F^i(\operatorname{coker} u) = F^iW/(F^iW \cap \operatorname{im} u)$.

We set $V = \mathbf{k}$ with two filtrations $F_1^iV = 0$ for $i \le 0$, $F_1^iV = \mathbf{k}$ for $i > 0$ and $F_2^iV = F_1^{i+1}V$. The identity map on V induces a morphism $u\colon (V, F_1^{\cdot}) \to (V, F_2^{\cdot})$. Then $\operatorname{coim}(u) = (V, F_1^{\cdot}) \not\simeq \operatorname{im}(u) = (V, F_2^{\cdot})$.

Lemma 5.4. *Let $f\colon X \to Y$ be a morphism in an abelian category. Then f is an isomorphism if and only if* $\ker f \simeq 0$ *and* $\operatorname{coker} f \simeq 0$.

Notation 5.5. Let $f\colon X \to Y$ be a morphism in an abelian category such that $\ker f \simeq 0$. We often write $Y/X := \operatorname{coker} f$.

Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence of morphisms in an abelian category. We assume that $g \circ f = 0$. Then there exists a natural morphism $a\colon \operatorname{im} f \to \ker g$. We have $\ker a \simeq 0$. We say that the sequence is exact (at Y) if this morphism is an isomorphism, that is, $\operatorname{coker} a \simeq 0$. A short sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is said exact if it is exact at X, Y and Z.

In this lecture we will argue with a general abelian category as if it were the category $\operatorname{Mod}(R)$ for some ring R. This is justified by *the Freyd-Mitchell embedding theorem*: for any "small" abelian category $\mathcal{C}$ there exists a fully faithful functor $\mathcal{C} \to \operatorname{Mod}(R)$ for some ring R (see [7] section 1.6). Without using this result we can also turn a "chase diagram proof" in $\operatorname{Mod}(R)$ into a proof for a general abelian category by using the following notion of "member" (for details see [3, §VIII.4]). For $A \in \mathcal{C}$, a *member* of A is a morphism $x\colon X \to A$; we write $x \in_m A$. We say that two members $x, y \in_m A$ are equivalent and write $x \equiv y$ if there exist epimorphisms $u\colon W \to X$, $v\colon W \to Y$ such that $x \circ u = y \circ v$. This is an equivalence relation. We have a list of properties of members similar to the expected properties of elements of a module over a ring. For example (i) $f\colon A \to B$ is a monomorphism if and only if for all $x \in_m A$, $f \circ x \equiv 0$ implies $x \equiv 0$; (ii) a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is exact at B if and only if $g \circ f = 0$ and for any $y \in_m B$ such that $g \circ y \equiv 0$ there exists $x \in_m A$ such that $f \circ x \equiv y$; … (see [3, §VIII.4, Thm 3]).

Lemma 5.6. *Let $\mathcal{C}$ be an abelian category and let*

$$\begin{array}{ccc} X & \xrightarrow{u} & X' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{v} & Y' \end{array}$$

be a commutative diagram in $\mathcal{C}$. Then there exist unique morphisms $\ker u \to \ker v$ *and* $\operatorname{coker} u \to \operatorname{coker} v$ *such that the following diagram commutes*

$$\begin{array}{ccccccc} \ker u & \longrightarrow & X & \xrightarrow{u} & X' & \longrightarrow & \operatorname{coker} u \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \downarrow \\ \ker v & \longrightarrow & Y & \xrightarrow{v} & Y' & \longrightarrow & \operatorname{coker} v. \end{array}$$

Let X be a topological space.

Proposition 5.7. *The category* $\mathbf{Psh}(X)$ *is abelian. Moreover, for* $u\colon P \to P'$ *in* $\mathbf{Psh}(X)$*, we have, for any open subset* $U \subset X$*,*

$$(\ker u)(U) \simeq \ker(u(U)\colon P(U) \to P'(U)),$$
$$(\operatorname{coker} u)(U) \simeq \operatorname{coker}(u(U)\colon P(U) \to P'(U)),$$

and, for all $x \in X$, $(\ker u)_x \simeq \ker(u_x)$ *and* $(\operatorname{coker} u)_x \simeq \operatorname{coker}(u_x)$*.*

Remark 5.8. Let $\mathcal{C}$ be a category and $\mathcal{C}'$ a full subcategory of $\mathcal{C}$, which means that $\mathrm{Ob}(\mathcal{C}')$ is a subset of $\mathrm{Ob}(\mathcal{C})$ and that $\mathrm{Hom}_{\mathcal{C}'}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$ for any $X, Y \in \mathcal{C}'$.

For $X, Y \in \mathcal{C}'$, if the product $X \times Y$ exists in $\mathcal{C}$ and belongs to $\mathcal{C}'$, then it is the product of X and Y in $\mathcal{C}'$. A similar remark holds for the coproduct, the kernel and the cokernel.

Proposition 5.9 (Back to associated sheaf)**.** *For any* $P \in \mathbf{Psh}(X)$*, there exist* $P^a \in \mathbf{Sh}(X)$ *and a morphism* $i\colon P \to P^a$ *which is universal in the sense: for any* $j\colon P \to F$*, with* $F \in \mathbf{Sh}(X)$*, there exists a unique* $f\colon P^a \to F$ *such that* $j = f \circ i$*:*

$$\begin{array}{ccc} P & \xrightarrow{\quad i \quad} & P^a \\ {\scriptstyle j}\downarrow & \swarrow{\scriptstyle f} & \\ F & & \end{array}$$

Moreover $i_x\colon P_x \to P^a_x$ *is an isomorphism, for each* $x \in X$*.*

Proposition 5.10. *The category* $\mathbf{Sh}(X)$ *is abelian. Moreover, for* $u\colon F \to F'$ *in* $\mathbf{Sh}(X)$*, we have, denoting by $\bar{u}$ the morphism u viewed in* $\mathbf{Psh}(X)$*,*

(a) $\ker \bar{u}$ *is a sheaf and* $\ker u \simeq \ker \bar{u}$*,*

(b) $\operatorname{coker} u \simeq (\operatorname{coker} \bar{u})^a$,
(c) *for all* $x \in X$, $(\ker u)_x \simeq \ker(u_x)$ *and* $(\operatorname{coker} u)_x \simeq \operatorname{coker}(u_x)$.

Lemma 5.11. *Let* X *be a topological space. If a sequence* $F \xrightarrow{u} G \xrightarrow{v} H$ *is exact in* $\mathbf{Psh}(X)$*, then the sequence* $F_x \xrightarrow{u_x} G_x \xrightarrow{v_x} H_x$ *is exact for each* $x \in X$.

A sequence $F \xrightarrow{u} G \xrightarrow{v} H$ *in* $\mathbf{Sh}(X)$ *is exact if and only if the sequence* $F_x \xrightarrow{u_x} G_x \xrightarrow{v_x} H_x$ *is exact for each* $x \in X$.

6. Exercises

Exercise 6.1. (My proof of Proposition 3.9 was a bit intricate. Here is a first step which can help.) Let X be a topological space, $P, Q \in \mathbf{Psh}(X)$ and let $f\colon P \to Q$ be a morphism. Let $x \in X$. We assume that $f_x\colon P_x \to Q_x$ is surjective. Let $U \in \mathrm{Op}(X)$ such that $x \in U$ and let $s \in Q(U)$. Prove that there exist a smaller open set $V \subset U$, with $x \in V$, and $t \in P(V)$ such that $f(V)(t) = s|_V$.

Exercise 6.2. (i) Let X be a topological space and A an abelian group. We define $P \in \mathbf{Psh}(X)$ as follows. For $U \in \mathrm{Op}(X)$, $P(U) = \{f\colon U \to A;\ f$ has a finite support$\}$. Here "finite support" means that $f(x) = 0$ except for finitely many x. The restriction maps $P(U) \to P(V)$ are defined as the usual restriction maps of functions, $f \mapsto f|_V$.

Prove that P is not a sheaf. Prove that $P_x \simeq A$ for all $x \in X$ (like the constant sheaf A_X).

(ii) We define $F \in \mathbf{Psh}(X)$ as follows. For $U \in \mathrm{Op}(X)$, $F(U) = \{f\colon U \to A;\ f$ has a locally finite support$\}$ (by this we mean that any $x \in U$ has a neighborhood V such that $f|_V$ has finite support). Prove that F is a sheaf. Prove that $F_x \simeq A$ for all $x \in X$. Prove that $F = P^a$.

Exercise 6.3. Let X be a topological space. Let $F, G \in \mathbf{Sh}(X)$ and $U \in \mathrm{Op}(X)$. We define $F|_U \in \mathbf{Sh}(U)$ by

$$F|_U(V) = F(V), \qquad \text{for } V \in \mathrm{Op}(U)$$

(check quickly that this defines indeed a sheaf). For $f\colon F \to G$ we also define $f|_U\colon F|_U \to G|_U$ by $f|_U(V) = f(V)$, for $V \in \mathrm{Op}(U)$. We obtain in this way a functor $\mathbf{Sh}(X) \to \mathbf{Sh}(U)$, $F \mapsto F|_U$.

Now we define a presheaf $\mathcal{H}om(F, G) \in \mathbf{Psh}(X)$ by

$$\mathcal{H}om(F, G)(U) = \mathrm{Hom}_{\mathbf{Sh}(U)}(F|_U, G|_U).$$

Prove that $\mathcal{H}om(F, G)$ is a sheaf.

Exercise 6.4. Let X be a topological space and A an abelian group. Let $U_1, U_2 \in \mathrm{Op}(X)$ such that $X = U_1 \cup U_2$. Let $F \in \mathbf{Sh}(X)$. We define $F|_{U_i} \in \mathbf{Sh}(U_i)$ as in Exercise 6.3. We assume that there exist isomorphisms $\varphi_i\colon F|_{U_i} \xrightarrow{\sim} A_{U_i}$, for $i = 1, 2$. We also assume that $U_1 \cap U_2$ is connected. Prove that $F \simeq A_X$. (For $U_1 \cap U_2$ not connected see Exercise 6.4.)

Exercise 6.5. (Notations of Exercises 6.3 6.4.) Let $X = \mathbb{C} \setminus \{0\}$. Let $\alpha \in \mathbb{C} \setminus \mathbb{Z}$. We consider the differential equation $(E) : z\frac{\partial f}{\partial z} = \alpha f$ (with local solution "$z \mapsto z^\alpha$"). For $U \in \mathrm{Op}(X)$ we set $F(U) = \{f\colon U \to \mathbb{C};$ f is a solution of $(E)\}$. Check that F is a sheaf. Find $U_1, U_2 \in \mathrm{Op}(X)$

such that $X = U_1 \cup U_2$ such that there exist isomorphisms $F|_{U_i} \simeq \mathbb{C}_{U_i}$, for $i = 1, 2$. Prove that $F \not\simeq \mathbb{C}_X$.

7. Exercises

Exercise 7.1. In an additive category $\mathcal{C}$, prove that f is an monomorphism if and only if the kernel of f exists and is 0. Prove that f is an epimorphism if and only if $\operatorname{coker} f \simeq 0$.

Exercise 7.2. Let $\mathcal{C}$ be an additive category and let $f\colon X \to Y$ be a morphism in $\mathcal{C}$. We assume that $\ker f$ exists. Prove that the morphism $i_f\colon \ker f \to X$ is a monomorphism.

Dually, if f has a cokernel, $p_f\colon Y \to \operatorname{coker} f$ is an epimorphism.

Exercise 7.3. Let $\mathcal{C}$ be an additive category and let $X, Y \in \mathcal{C}$. Since $\operatorname{Hom}(X, X \times Y) = \operatorname{Hom}(X, X) \times \operatorname{Hom}(X, Y)$, $(\mathrm{id}_X, 0)$ gives a morphism $i_X\colon X \to X \times Y$. In the same way we define $i_Y\colon Y \to X \times Y$.

We also have $\operatorname{Hom}(X \oplus Y, Z) = \operatorname{Hom}(X, Z) \times \operatorname{Hom}(X, Z)$ and (i_X, i_Y) defines $u\colon X \oplus Y \to X \times Y$. Prove that u is an isomorphism.

Exercise 7.4. Let X be a topological space, and $Z \subset X$ a closed subset. Let A be an abelian group. We define the presheaf $PA_{X,Z}$ on X by $PA_{X,Z}(U) = 0$ if $Z \cap U = \emptyset$ and $PA_{X,Z}(U) = A$ if $Z \cap U \neq \emptyset$ with the restriction maps $r_{V,U} = \mathrm{id}_A$ if $V \cap Z \neq \emptyset$ (otherwise $r_{V,U}$ must be 0).

We set $A_{X,Z} = (PA_{X,Z})^a$. Using the construction of P^a in the proof of Proposition 3.10 check that $A_{X,Z}(U) = \{f\colon U \cap Z \to A;\ f \text{ is locally constant}\}$. (Here, locally constant means: any $x \in U \cap Z$ has a neighborhood $V(x)$ such that $f|_{V(x)}$ is constant.) In particular, if X is locally connected, then $A_{X,Z}(U) \simeq A^{\pi_0(Z \cap U)}$, where $\pi_0(Z \cap U)$ is the set of connected components of $Z \cap U$. Note that, when X is not locally connected, for example $X = \mathbb{Q}$, this last description does not hold.

Check that $(A_{X,Z})_x \simeq A$ if $x \in Z$ and $(A_{X,Z})_x \simeq 0$ otherwise.

Exercise 7.5. In the previous exercise we could try to remove the condition "Z is closed" but the result is more complicated. Set $A'_{X,Z} = (PA_{X,Z})^a$ for a general Z. Describe $A'_{X,Z}$ if we take (1) $X = \mathbb{R}^n$, Z an open ball in $\mathbb{R}^n$ (2) $X = \mathbb{R}$, $Z = \mathbb{R} \setminus \{0\}$. In particular the stalks $(A'_{X,Z})_x$ don't satisfy the same relation as in the previous exercise.

Exercise 7.6. Let X be a topological space and $Z \subset X$ a closed subset. Let A be an abelian group. Recall the "constant sheaf on Z" $A_{X,Z}$, such that $A_{X,Z}(U) = \{f\colon U \cap Z \to A;\ f \text{ is locally constant}\}$.

Let $Z' \subset Z$ be a closed subset. We define a morphism $res^Z_{Z'}\colon A_{X,Z} \to A_{X,Z'}$ by $r(U)(f) = f|_{U \cap Z'}$ (check that this is a sheaf morphism). For an open subset $U \subset X$ we define

$$A_{X,U} = \ker(r^X_{X \setminus U}\colon A_{X,X} \to A_{X,X \setminus U}).$$

Verify that $(A_{X,U})_x \simeq \begin{cases} A & \text{if } x \in U, \\ 0 & \text{if } x \notin U. \end{cases}$

Exercise 7.7. Let X be a topological space, $U \subset X$ an open subset and $Z = X \setminus U$. We set $F = A_{X,U} \oplus A_{X,Z}$. Check that there exist two exact sequences: $0 \to A_{X,U} \to F \to A_{X,Z} \to 0$ and $0 \to A_{X,U} \to A_{X,X} \to A_{X,Z} \to 0$.

Check that $F_x \simeq (A_{X,X})_x$ for all $x \in X$.

We consider $X = \mathbb{R}$, $U =]-\infty, 0[$ and $Z = [0, +\infty[$. Prove that $\mathrm{Hom}(A_{X,Z}, A_{X,X}) = 0$.

With F as above, prove that $F \not\simeq A_{X,X}$.

Exercise 7.8. Let $\mathcal{C}$ be an abelian category and let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be two morphisms in $\mathcal{C}$. We assume that $\ker(g \circ f) = 0$. Prove that $\ker(f) = 0$.

Exercise 7.9. Let $\mathcal{C}$ be a category. We define the category of morphisms in $\mathcal{C}$, say $\mathrm{Mor}(\mathcal{C})$, as the category whose objects are the morphisms in $\mathcal{C}$ (that is an object is the data of $X \xrightarrow{u} X'$) and the morphisms are the commutative diagrams $\mathrm{Hom}_{\mathrm{Mor}(\mathcal{C})}((X \xrightarrow{u} X'), (Y \xrightarrow{v} Y')) = \{(f, f'); f \colon X \to Y, f' \colon X' \to Y', v \circ f = f' \circ u\}$. The composition is given termwise by the composition in $\mathcal{C}$, that is, $(g, g') \circ (f, f') = (g \circ f, g' \circ f')$.

We assume that $\mathcal{C}$ is abelian. Prove that $\mathrm{Mor}(\mathcal{C})$ is also abelian. (To save time, you can admit that $\mathrm{Mor}(\mathcal{C})$ is additive and check only the existence of kernels, cokernels and Definition 5.2 using Lemma 5.6.)

Exercise 7.10. We let $\mathcal{O}_\mathbb{C} \in \mathbf{Sh}(\mathbb{C})$ be the sheaf of holomorphic functions over $\mathbb{C}$, that is, $\mathcal{O}_\mathbb{C}(U) = \{f \colon U \to \mathbb{C};\ f \text{ is holomorphic}\}$. We let $\mathcal{O}_\mathbb{C}^\times$ be the sheaf of non vanishing holomorphic functions and we denote by $\exp \colon \mathcal{O}_\mathbb{C} \to \mathcal{O}_\mathbb{C}^\times$ the morphism $f \mapsto \exp(f)$. Prove that we have an exact sequence $0 \to \mathbb{Z}_\mathbb{C} \to \mathcal{O}_\mathbb{C} \xrightarrow{\exp} \mathcal{O}_\mathbb{C}^\times \to 0$ in $\mathbf{Sh}(\mathbb{C})$.

Prove that this sequence is not exact in $\mathbf{Psh}(\mathbb{C})$.

Exercise 7.11. (Variation on Exercise 7.10) We keep the notations of Exercise 7.10 Let $u \colon \mathcal{O}_\mathbb{C} \to \mathcal{O}_\mathbb{C}$ be the derivation, that is, $u(U)(f) = f'$ for $f \in \mathcal{O}_\mathbb{C}(U)$. What is $\ker u$, $\mathrm{coker}\, u$ in $\mathbf{Sh}(\mathbb{C})$? Prove that u is not surjective in $\mathbf{Psh}(\mathbb{C})$.

Exercise 7.12. Inductive limit (also called "colimit"). We give a definition in the special case of a filtrant indexing set. Let $(\mathcal{I}, \leq)$ be an ordered set which is *filtrant*, which means: for any $i, j \in \mathcal{I}$ there exists $k \in \mathcal{I}$ such that $i \leq k$ and $j \leq k$. Typical examples are $\mathcal{I} = \mathbb{N}$

and, for a topological space X and a point $x \in X$, $\mathcal{I}$ is the set of open neighborhoods of x.

Let $\{E_i, u_{ji}\}$ be an inductive system of sets indexed by $\mathcal{I}$, which means: u_{ji} is a map $u_{ji}\colon E_i \to E_j$ for any $i \leq j$ such that $u_{ii} = \mathrm{id}_{E_i}$ and $u_{kj} \circ u_{ji} = u_{ki}$ when $i \leq j \leq k$. Then

$$\varinjlim_{i \in \mathcal{I}} E_i = \bigsqcup_{i \in \mathcal{I}} E_i / \sim,$$

where $\sim$ is the equivalence relation defined by $x_i \in E_i \ \sim \ x_j \in E_j$ if there exists k with $i, j \leq k$ and $u_{ki}(x_i) = u_{kj}(x_j)$. This set comes with natural maps $u_i\colon E_i \to \varinjlim_{i \in \mathcal{I}} E_i$ induced by the inclusion of E_i in $\bigsqcup E_k$. We remark that any element of $\varinjlim_{i \in \mathcal{I}} E_i$ is represented by an element $x_{i_0} \in E_{i_0}$ for some $i_0 \in \mathcal{I}$.

(1) Check that, if the E_i are groups and the u_{ji} are group morphisms, then $\varinjlim_{i \in \mathcal{I}} E_i$ has a unique group structure such that the maps u_i are group morphisms.

(2) When $\mathcal{I} = \mathbb{N}$ we only need to specify the maps $u_{i+1,i}$. Take $E_i = \mathbb{Z}$ for all i and $u_{i+1,i}(x) = 2x$ for all i. We write for short $\varinjlim_{i \in \mathbb{N}} E_i = \varinjlim(\mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \xrightarrow{2\cdot} \cdots)$. What is this colimit ? (Identify with a subgroup of $\mathbb{Q}$.)

(3) Give an example of an inductive system of groups indexed by $\mathbb{N}$, $E_0 \xrightarrow{u_{1,0}} E_1 \to \cdots$, where all groups and all maps $u_{i+1,i}$ are non zero, but $\varinjlim_{i \in \mathbb{N}} E_i \simeq 0$.

(4) Let P be a presheaf on a topological space X and $x \in X$. We assume that x has a countable system of decreasing open neighborhoods B_n. (Typically $X = \mathbb{R}^n$ and B_n is the open ball with center x and radius $1/n$.) Check that $P_x \simeq \varinjlim_{n \in \mathbb{N}} P(B_n)$.

Exercise 7.13. We keep the framework of Exercise 7.12. Let $\mathcal{I}$ be a filtrant ordered set. Let $\{E_i, u_{ji}\}$, $\{F_i, v_{ji}\}$ be inductive systems indexed by $\mathcal{I}$. We remark that $\varinjlim_{i \in \mathcal{I}} E_i$ comes with maps $\pi_i\colon E_i \to \varinjlim_{i \in \mathcal{I}} E_i$ (we use abusively the same notation for the F_i's). We assume to be given maps $f_i\colon E_i \to F_i$ commuting with the u_{ji}, v_{ji}.

Check that these maps induce a unique map $f\colon \varinjlim_{i \in \mathcal{I}} E_i \to \varinjlim_{i \in \mathcal{I}} F_i$ such that $f \circ \pi_i = \pi_i \circ f_i$ for all i.

Now we assume that our inductive systems are made of abelian groups and all maps are additive. Then $\varinjlim_{i \in \mathcal{I}} E_i$ is an abelian group. We remark that u_{ji} maps $\ker(f_i)$ to $\ker(f_j)$ and we obtain an inductive system $\{\ker(f_i), u_{ji}\}$. Check that $\varinjlim_{i \in \mathcal{I}} \ker(f_i) \simeq \ker(f)$.

In the same way check that $\varinjlim_{i \in \mathcal{I}} \mathrm{coker}(f_i) \simeq \mathrm{coker}(f)$.

8. Homework

Exercise 8.1. Let $\mathbf{k}$ be a commutative field and $\mathcal{V}$ the category of $\mathbf{k}$-vector spaces. Let $\mathrm{id}_{\mathcal{V}}$ be the identity functor of $\mathcal{V}$. For $a \in \mathbf{k}$ we define a morphism of functors $\theta_a\colon \mathrm{id}_{\mathcal{V}} \to \mathrm{id}_{\mathcal{V}}$ by $\theta_a(E) = a\,\mathrm{id}_E$ for every vector space E. Prove that this gives all morphisms from $\mathrm{id}_{\mathcal{V}}$ to itself:

$$\mathrm{Hom}_{\mathrm{Fct}(\mathcal{V},\mathcal{V})}(\mathrm{id}_{\mathcal{V}}, \mathrm{id}_{\mathcal{V}}) = \mathbf{k}.$$

Exercise 8.2. Let $\mathbf{Ab}_f$ be the category of **finite** abelian groups. For $G \in \mathbf{Ab}_f$ a *character* of G is a group morphism $\chi\colon G \to \mathbb{C}^*$ (with $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ the multiplicative group). We define the group of characters $\widehat{G} = \mathrm{Hom}_{\mathbf{Ab}}(G, \mathbb{C}^*)$, where the product is defined by $(\chi_1\chi_2)(g) = \chi_1(g)\chi_2(g)$. Prove that $\widehat{\mathbb{Z}/n\mathbb{Z}} \simeq \mathbb{Z}/n\mathbb{Z}$. Prove that $\widehat{G_1 \times G_2} \simeq \widehat{G}_1 \times \widehat{G}_2$ for $G_1, G_2 \in \mathbf{Ab}_f$. Since any $G \in \mathbf{Ab}_f$ is decomposed as a product of cyclic groups, this proves that there exists an isomorphism $G \simeq \widehat{G}$. Deduce that $\mathbf{Ab}_f^{\mathrm{op}}$ is equivalent to $\mathbf{Ab}_f$.

Exercise 8.3. Let $\mathcal{C}$ be the category $\mathrm{Mod}(R)$ for some ring R (we can take $\mathcal{C} = \mathbf{Ab}$). We define the category of morphisms in $\mathcal{C}$, say $\mathrm{Mor}(\mathcal{C})$, as the category whose objects are the morphisms in $\mathcal{C}$ (that is an object is the data of $X \xrightarrow{f} Y$) and the morphisms are the commutative diagrams

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{Mor}(\mathcal{C})}((X \xrightarrow{f} Y), (X' \xrightarrow{f'} Y')) \\ &\qquad = \{(u, v);\ u\colon X \to X',\ v\colon Y \to Y', v \circ f = f' \circ u\}\end{aligned}$$

illustrated by
$$\begin{array}{ccc} X & \xrightarrow{u} & X' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{v} & Y' \end{array}$$

The composition is given termwise by the composition in $\mathcal{C}$, that is, $(u_2, v_2) \circ (u_1, v_1) = (u_2 \circ u_1, v_2 \circ v_1)$.

We admit that $\mathrm{Mor}(\mathcal{C})$ is abelian with sums, kernels, cokernels given termwise (for example with the above notations $\ker(u, v) = (\ker(u) \xrightarrow{\bar{f}} \ker(v))$ where $\bar{f}$ is induced by f – see Lemma 5.6).

(i) Let $(P \xrightarrow{a} Q)$ be a projective object in $\mathrm{Mor}(\mathcal{C})$. Prove that P and Q must be projective in $\mathcal{C}$. Prove that there exists $b\colon Q \to P$ such that $b \circ a = \mathrm{id}_P$.

(ii) Let P be a projective object in $\mathcal{C}$. Prove that $(0 \to P)$ and $(P \xrightarrow{\mathrm{id}} P)$ are projective in $\mathrm{Mor}(\mathcal{C})$.

(iii) Prove that there are "enough projectives" in $\mathrm{Mor}(\mathcal{C})$ in the sense that, for any $(X \xrightarrow{f} Y)$ there exists $(P \xrightarrow{a} Q)$ a projective object in $\mathrm{Mor}(\mathcal{C})$ and an epimorphism
$$\begin{array}{ccccc} P & \xrightarrow{u} & X & \longrightarrow & 0 \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle f} & & \\ Q & \xrightarrow{v} & Y & \longrightarrow & 0 \end{array}$$

Exercise 8.4. Let $\mathbf{k}$ be a ring. Let $W \subset \mathbb{R}^n$ be a subset. For $U \in \mathrm{Op}(\mathbb{R}^n)$ we define

$$\mathbf{k}_W(U) = \{f : W \cap U \to \mathbf{k};\ f \text{ is locally constant and } \operatorname{supp}(f) \text{ is closed in } U\},$$

where $\operatorname{supp}(f) = \{x \in W \cap U;\ f(x) \neq 0\}$.

(i) For $n = 1$, $W = [0, 2[$ (or $[0, 2)$ in english notation), what is $\mathbf{k}_W(]-1, 1[)$, $\mathbf{k}_W(]0, 2[)$, $\mathbf{k}_W(]1, 3[)$?

(ii) For $U' \subset U$, U' open, check that the restriction of functions $f \mapsto f|_{U'}$ gives a map $\mathbf{k}_W(U) \to \mathbf{k}_W(U')$. Prove that $\mathbf{k}_W$ is a sheaf.

(iii) We say that W is "locally closed" if we can write $W = V \cap Z$, with V open and Z closed. In this case prove that

$$(\mathbf{k}_W)_x \simeq \begin{cases} \mathbf{k} & \text{if } x \in W, \\ 0 & \text{if } x \notin W. \end{cases}$$

(iv) For $n = 2$, we consider $W = [0, +\infty[\times]0, +\infty[\cup \{(0, 0)\}$. What is $(\mathbf{k}_W)_{(0,0)}$?

(v) Let $W \subset \mathbb{R}^n$ be locally closed. Let $W_1 \subset W$ be closed in W. For $U \in \mathrm{Op}(\mathbb{R}^n)$ check that the restriction of functions $f \mapsto f|_{U \cap W_1}$ gives a map $\mathbf{k}_W(U) \to \mathbf{k}_{W_1}(U)$ and that this gives a morphism of sheaves $\mathbf{k}_W \to \mathbf{k}_{W_1}$.

(vi) For $n = 1$, we set $I = [0, 3]$, $J =]1, 2[$. Prove that there is no non-zero morphism of sheaves $\mathbf{k}_I \to \mathbf{k}_J$.

9. Correction of the homework

Exercise 9.1. (i) $\theta_a \colon \mathrm{id}_{\mathcal{V}} \to \mathrm{id}_{\mathcal{V}}$ is a morphism of functors: For $E \in \mathcal{V}$ we have $\mathrm{id}_{\mathcal{V}}(E) = E$. So we have to check that, for any morphism $f \colon E \to F$ in $\mathcal{V}$ the diagram

$$\begin{array}{ccc} E & \xrightarrow{a\,\mathrm{id}_E} & E \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ F & \xrightarrow[a\,\mathrm{id}_F]{} & F. \end{array}$$

is commutative. This follows from the linearity of the map f.

(ii) Conversely, let $\theta \colon \mathrm{id}_{\mathcal{V}} \to \mathrm{id}_{\mathcal{V}}$ be a morphism of functors. For $E = \mathbf{k}$, $\theta(\mathbf{k})$ is a linear map from $\mathbf{k}$ to $\mathbf{k}$. Such a map is a multiplication by a scalar. So $\theta(\mathbf{k}) = a\,\mathrm{id}_{\mathbf{k}}$, for some $a \in \mathbf{k}$. Now let $E \in \mathcal{V}$ and let $x \in E$. We define $f \colon \mathbf{k} \to E$ by $f(t) = tx$. Since θ is a morphism of functors, we have the commutative diagram

$$\begin{array}{ccc} \mathbf{k} & \xrightarrow{a\,\mathrm{id}_{\mathbf{k}}} & \mathbf{k} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ E & \xrightarrow[\theta(E)]{} & E. \end{array}$$

It follows that $(\theta(E))(x) = ax$. This holds for any $x \in E$, so $\theta(E) = a\mathrm{id}_E$. Finally $\theta = \theta_a$, as required.

Exercise 9.2. (i) We remark that a group morphism $\chi \colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^*$ is determined by $z = \chi(1)$. Conversely, $z \in \mathbb{C}^*$ defines a group morphism $\chi \colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^*$ with $z = \chi(1)$ if and only if $z^n = 1$. Let $U_n \subset \mathbb{C}^*$ be the group of n^{th}-roots of 1. Then $\chi \mapsto \chi(1)$ gives a bijection $a_n \colon \widehat{\mathbb{Z}/n\mathbb{Z}} \to U_n$. Since the group structure on $\widehat{\mathbb{Z}/n\mathbb{Z}}$ is induced by the multiplication on $\mathbb{C}^*$, the map a_n is a group morphism. Hence $\widehat{\mathbb{Z}/n\mathbb{Z}} \simeq U_n$. Since $U_n \simeq \mathbb{Z}/n\mathbb{Z}$, this proves the result.

(ii) Let $i_1 \colon G_1 \to G_1 \times G_2$, $g \mapsto (g, 0)$, and $i_2 \colon G_2 \to G_1 \times G_2$, $g \mapsto (0, g)$. Then i_1, i_2 are group morphisms. We define a first map $a \colon \widehat{G_1 \times G_2} \to \widehat{G}_1 \times \widehat{G}_2$ by $a(\chi) = (\chi \circ i_1, \chi \circ i_2)$ (we note that $\chi \circ i_k$ is a group morphism since it is the composition of two morphisms). For $\chi, \chi' \in \widehat{G_1 \times G_2}$, we have $(\chi \cdot \chi') \circ i_k = (\chi \circ i_k) \cdot (\chi' \circ i_k)$. Hence a is a group morphism.

Similarly, we define $p_1 \colon G_1 \times G_2 \to G_1$ and $p_2 \colon G_1 \times G_2 \to G_2$ by $p_k(g_1, g_2) = g_k$. Then p_1, p_2 are group morphisms, hence $(\chi_1 \circ p_1)$ and $(\chi_2 \circ p_2)$ also, and we can define $b \colon \widehat{G}_1 \times \widehat{G}_2 \to \widehat{G_1 \times G_2}$ by $b(\chi_1, \chi_2) = (\chi_1 \circ p_1) \cdot (\chi_2 \circ p_2)$.

Now we have $(b\circ a)(\chi) = b(\chi\circ i_1, \chi\circ i_2) = (\chi\circ i_1\circ p_1)\cdot(\chi\circ i_2\circ p_2)$, hence $((b\circ a)(\chi))(g_1, g_2) = \chi(g_1, 1)\cdot\chi(1, g_2) = \chi((g_1, 1)\cdot(1, g_2)) = \chi(g_1, g_2)$. Hence $b\circ a = \mathrm{id}$.

We also have $(a\circ b)(\chi_1, \chi_2) = (\chi_1\circ p_1\circ i_1, \chi_2\circ p_2\circ i_2) = (\chi_1, \chi_2)$. Hence $a\circ b = \mathrm{id}$.

This proves that a is also bijective and $\widehat{G_1\times G_2} \simeq \widehat{G}_1 \times \widehat{G}_2$.

(iii) We remark that $G\mapsto \widehat{G}$ reverses the composition of maps: a group morphism $u\colon G\to H$ gives a morphism $\hat{u}\colon \widehat{H}\to\widehat{G}$ by $\hat{u}(\chi) = \chi\circ u$. Then we have $\widehat{u\circ v} = \hat{v}\circ\hat{u}$. So we obtain a functor $\mathcal{F}\colon \mathbf{Ab}\to\mathbf{Ab}^{\mathrm{op}}$, $G\mapsto\mathcal{F}(G) := \widehat{G}$.

We claim that $\mathcal{F}\circ\mathcal{F}\simeq\mathrm{id}$.

Indeed we have a morphism $\mathrm{ev}_G\colon G\to\widehat{\widehat{G}}$ for any abelian group G defined by $\mathrm{ev}_G(g)\colon \widehat{G}\to\mathbb{C}^*$, $(\mathrm{ev}_G(g))(\chi) = \chi(g)$. If G is finite, the morphism ev_G is injective since, for any $g\neq 0\in G$ we can find $\chi\in\widehat{G}$ such that $\chi(g)\neq 1$: decomposing G as a product of cyclic groups we can assume G is cyclic, say $G = \mathbb{Z}/n\mathbb{Z}$, in which case the character $\chi_0\colon G\to\mathbb{C}^*$, $[p]\mapsto exp(2i\pi p/n)$, is injective.

Since ev_G is injective and $|G| = |\widehat{\widehat{G}}|$ (because G and $\widehat{\widehat{G}}$ are isomorphic), we see that ev_G is an isomorphism.

Hence $\mathrm{ev}\colon \mathrm{id}\to\mathcal{F}\circ\mathcal{F}$, defined by $\mathrm{ev}(G) = \mathrm{ev}_G$, is an isomorphism of functors. This proves that $\mathbf{Ab}\xrightarrow{\sim}\mathbf{Ab}^{\mathrm{op}}$.

Exercise 9.3. (i-a) We prove that P is projective. Let $u\colon X\twoheadrightarrow X'$ be an epimorphism in $\mathcal{C}$ and $v\colon P\to X'$ be any morphism. We consider

$$\tilde{u} = \begin{array}{ccc} X & \xrightarrow{u} & X' \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{array} \qquad \tilde{v} = \begin{array}{ccc} P & \xrightarrow{v} & X' \\ \downarrow & & \downarrow \\ Q & \longrightarrow & 0 \end{array}$$

We know that there exists $\tilde{w}$ such that $\tilde{v} = \tilde{u}\circ\tilde{w}$. Writing $\tilde{w} = (w, 0)$ we see that w satisfies $v = u\circ w$, as required.

(i-b) We prove that Q is projective. Let $u\colon Y\twoheadrightarrow Y'$ be an epimorphism in $\mathcal{C}$ and $v\colon Q\to Y'$ be any morphism. We consider

$$\tilde{u} = \begin{array}{ccc} Y & \xrightarrow{u} & Y' \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\ Y & \xrightarrow{u} & Y' \end{array} \qquad \tilde{v} = \begin{array}{ccc} P & \xrightarrow{v\circ a} & Y' \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle \mathrm{id}} \\ Q & \xrightarrow{v} & Y' \end{array}$$

Again there exists $\tilde{w} = (w_1, w_2)$ such that $\tilde{v} = \tilde{u}\circ\tilde{w}$. We have in particular $v = u\circ w_2$, as required.

(i-c) We have an epimorphism $\tilde{u} = (\mathrm{id}_P, 0)\colon (P \to P) \twoheadrightarrow (P \to 0)$ and a morphism $\tilde{v} = (\mathrm{id}_P, 0)\colon (P \to Q) \to (P \to 0)$:

$$\tilde{u} = \begin{array}{ccc} P & \xrightarrow{\mathrm{id}_P} & P \\ \downarrow{\scriptstyle \mathrm{id}_P} & & \downarrow \\ P & \longrightarrow & 0 \end{array} \qquad \tilde{v} = \begin{array}{ccc} P & \xrightarrow{\mathrm{id}_P} & P \\ \downarrow{\scriptstyle a} & & \downarrow \\ Q & \longrightarrow & 0 \end{array}$$

We can factorize $\tilde{v}$ through $\tilde{w} = (w_1, w_2)\colon (P \to Q) \to (P \to P)$

$$\tilde{w} = \begin{array}{ccc} P & \xrightarrow{w_1} & P \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle \mathrm{id}_P} \\ Q & \xrightarrow{w_2} & P \end{array}$$

such that $\tilde{v} = \tilde{u} \circ \tilde{w}$. We obtain $w_2 \circ a = \mathrm{id}_P \circ w_1$ and $\mathrm{id}_P = \mathrm{id}_P \circ w_1$. Hence $w_1 = \mathrm{id}_P$ and $w_2 \circ a = \mathrm{id}_P$. We can take $b = w_2$.

(ii) We pick an epimorphism $\tilde{u} = (u_1, u_2)\colon (X \xrightarrow{f} Y) \twoheadrightarrow (X' \xrightarrow{f'} Y')$ and a morphism $\tilde{v} = (0, v_2)\colon (0 \to P) \twoheadrightarrow (X' \xrightarrow{f'} Y')$:

$$\begin{array}{ccc} X & \xrightarrow{u_1} & X' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{u_2} & Y' \end{array} \qquad \begin{array}{ccc} 0 & \xrightarrow{0} & X' \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle f'} \\ P & \xrightarrow{v_2} & Y' \end{array}$$

Since P is projective, we can factorize v_2 through $w_2\colon P \to Y$ and we set $\tilde{w} = (0, w_2)$. Then $\tilde{v} = \tilde{u} \circ \tilde{w}$ as required.

For the case $(P \xrightarrow{id} P)$ we use the notations:

$$\tilde{u} = \begin{array}{ccc} X & \xrightarrow{u_1} & X' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{u_2} & Y' \end{array} \qquad \tilde{v} = \begin{array}{ccc} P & \xrightarrow{v_1} & X' \\ \downarrow{\scriptstyle \mathrm{id}_P} & & \downarrow{\scriptstyle f'} \\ P & \xrightarrow{v_2} & Y' \end{array}$$

We must have $v_2 = f' \circ v_1$. Since P is projective, we can factorize v_1 through $w_1\colon P \to X$. We set $\tilde{w} = \begin{array}{ccc} P & \xrightarrow{w_1} & X \\ \downarrow{\scriptstyle \mathrm{id}_P} & & \downarrow{\scriptstyle f} \\ P & \xrightarrow{f \circ w_1} & Y \end{array}$. Then $\tilde{v} = \tilde{u} \circ \tilde{w}$.

(iii) It is a general fact that the sum of two projectives P, P' is projective, because $\mathrm{Hom}(P \oplus P', X) = \mathrm{Hom}(P, X) \oplus \mathrm{Hom}(P', X)$ for any X. Since $\mathrm{Mod}(R)$ has enough projectives we can find projectives P, Q and

epimorphisms $u_0\colon P \twoheadrightarrow X$, $v_0\colon Q \twoheadrightarrow Y$. We define $\begin{array}{ccc} P & \xrightarrow{u} & X \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle f} \\ P \oplus Q & \xrightarrow{v} & Y \end{array}$
by $a = (\mathrm{id}_P, 0)$, $u = u_0$, $v = (f \circ u_0, v_0)$. The diagram commutes, u, v are epimorphisms and $(P \xrightarrow{a} P \oplus Q) = (P \xrightarrow{\mathrm{id}_P} P) \oplus (0 \to Q)$ is projective, as required.

Exercise 9.4. (i-a) By definition $\mathbf{k}_W(]-1,1[) = \{f\colon [0,1[\to \mathbf{k};\ f$ is locally constant and $\mathrm{supp}(f)$ is closed in $]-1,1[\}$. Since $[0,1[$ is locally connected and connected, f is constant on $[0,1[$. Then its support is empty or $[0,1[$ and in both cases it is closed in $]-1,1[$. Hence $\mathbf{k}_W(]-1,1[) \simeq \mathbf{k}$.

(i-b) The case $\mathbf{k}_W(]0,2[)$ is completely similar, replacing $]-1,1[$ by $]0,2[$. We find $\mathbf{k}_W(]0,2[) \simeq \mathbf{k}$.

(i-c) By definition $\mathbf{k}_W(]1,3[) = \{f\colon]1,2[\to \mathbf{k};\ f$ is locally constant and $\mathrm{supp}(f)$ is closed in $]1,3[\}$. Again f is constant on $]1,2[$. Hence, if $f \neq 0$, its support is $]1,2[$, which is not closed in $]1,3[$. So we must have $f = 0$: $\mathbf{k}_W(]1,3[) = 0$.

(ii) Let $f \in \mathbf{k}_W(U)$, that is, $f\colon U \cap W \to \mathbf{k}$ locally constant with $\mathrm{supp}(f)$ closed in U. So we can write $\mathrm{supp}(f) = Z \cap U$, with Z closed in $\mathbb{R}^n$. Then $f' = f|_{W\cap U'}$ is still locally constant and we have: $\mathrm{supp}(f') = U' \cap \mathrm{supp}(f) = U' \cap (U \cap Z) = U' \cap Z$ is closed. Hence f' belongs to $\mathbf{k}_W(U')$, as required.

For $U'' \subset U' \subset U$ it is clear that $(f|_{W\cap U'})|_{W\cap U''} = f|_{W\cap U''}$. Hence $\mathbf{k}_W$ is a presheaf. Now assume U has a covering $U = \bigcup_{i\in I} U_i$. The "separation" property holds because it holds for functions. For the "gluing" property, let $f_i \in \mathbf{k}_W(U_i)$ be functions which coincide on the double intersections $U_i \cap U_j$. We can glue the f_i's as a function $f\colon U \cap W \to \mathbf{k}$. Then f is locally constant and we have to check the support condition. We set $V = U \setminus \mathrm{supp}(f)$ and $V_i = U_i \setminus \mathrm{supp}(f_i)$. Since V_i is open in U_i and U_i is open in $\mathbb{R}^n$, V_i is also open in $\mathbb{R}^n$. Since $f|_{U_i} = f_i$ we have $V \cap U_i = V_i$. Hence $V = \bigcup_i (V \cap U_i) = \bigcup_i V_i$ is open. Hence $\mathrm{supp}(f)$ is closed in U.

(iii-a) We first assume $x \in W$. For an open set U with $x \in U$ we define $\mathrm{ev}_x^U\colon \mathbf{k}_W(U) \to \mathbf{k}$ by $f \mapsto f(x)$. If $U' \subset U$, we have of course $f|_{U'\cap W}(x) = f(x)$. Hence $\mathrm{ev}_x^{U'}(f|_{U'}) = \mathrm{ev}_x^U(f)$. It follows that the maps ev_x^U, for U running over the open neighborhoods of x are compatible and define a map $\mathrm{ev}_x\colon (\mathbf{k}_W)_x = \varinjlim_{U,\, x\in U} \mathbf{k}_W(U) \to \mathbf{k}$. This map is clearly $\mathbf{k}$-linear.

Let us prove that u is bijective. First assume that $\mathrm{ev}_x(g) = 0$ for some $g \in (\mathbf{k}_W)_x$. The germ g is represented by $f \in \mathbf{k}_W(U)$ for some open neighborhood U of x. Then we have $\mathrm{ev}_x(g) = \mathrm{ev}^U_x(f) = f(x)$. Then $f(x) = 0$. Since f is locally constant we can find some neighborhood U' of x, with $U' \subset U$, such that $f \equiv 0$ on U'. But $f|_{U'}$ also represents g. Hence $g = 0$, as required.

Now we prove that ev_x is surjective. Since the target is $\mathbf{k}$ it is enough to find a preimage for $1 \in \mathbf{k}$. We can write $W = V \cap Z$, V open, Z closed. In particular W is closed in V. We define $f\colon W \cap V = W \to \mathbf{k}$, by $f(x) = 1$ for all $x \in W$. Then $\mathrm{supp}(f) = W$ is closed in V and f defines a section $f \in \mathbf{k}_W(V)$. Since $x \in W \subset V$, V is an open neighborhood of x, and f defines a germ $g = [f] \in (\mathbf{k}_W)_x$. We have $\mathrm{ev}_x(g) = \mathrm{ev}^V_x(f) = f(x) = 1$, as required.

(iii-b) We assume $x \notin W$. We pick a germ $g \in (\mathbf{k}_W)_x$, represented by $f \in \mathbf{k}_W(U)$ for some open neighborhood U of x. By definition we have $\mathrm{supp}(f) \subset W$ and $\mathrm{supp}(f)$ closed in U. Since $x \notin W$, we have $x \notin \mathrm{supp}(f)$. Since $\mathrm{supp}(f)$ is closed in U, we can find a smaller open neighborhood of x, say $U' \subset U$, such that $U' \cap \mathrm{supp}(f) = \emptyset$. Then $f|_{W\cap U'} = 0$. But $f|_{W\cap U'}$ is also a representative of g. Hence $g = 0$. Finally $(\mathbf{k}_W)_x = 0$.

(iv) We set $x = (0, 0)$. We pick a germ $g \in (\mathbf{k}_W)_x$, represented by $f \in \mathbf{k}_W(U)$ for some open neighborhood U of x. Since $f\colon W \cap U \to \mathbf{k}$ is locally constant, we can find a small disc $D = D(x, \varepsilon)$, $\varepsilon > 0$, such that $D \subset U$ and $f|_{W\cap D}$ is constant, say with value c. If $c \neq 0$, we then have $\mathrm{supp}(f) \cap D = W \cap D$. But $W \cap D$ is not closed in D and we have a contradiction. Hence $c = 0$ and f vanishes in a neighborhood of x. It follows as in (iii-b) that $g = 0$. Finally $(\mathbf{k}_W)_x = 0$.

(v) Let $f \in \mathbf{k}_W(U)$, $f\colon W \cap U \to \mathbf{k}$. We have to check that $f_1 = f|_{W_1\cap U}\colon W_1 \cap U \to \mathbf{k}$ is locally constant and $\mathrm{supp}(f_1)$ is closed in U. If f is constant on $W \cap V$ for some open susbet $V \subset U$, then f_1 is constant on $W_1 \cap V$. Hence f_1 is locally constant. We also have $\mathrm{supp}(f_1) = W_1 \cap \mathrm{supp}(f)$. Since W_1 is closed in W, we can write $W_1 = W \cap Z_1$, with Z_1 closed in $\mathbb{R}^n$. We can also write $\mathrm{supp}(f) = Z_2 \cap U$, with Z_2 closed in $\mathbb{R}^n$. Then $\mathrm{supp}(f_1) = Z_1 \cap Z_2 \cap U$ is closed in U.

For an open subset $U' \subset U$, we have $(f|_{W_1\cap U})|_{U'} = (f|_{U'})|_{W_1\cap U'}$. This proves that $f \mapsto f|_{W_1\cap U}$ defines a morphism of sheaves.

(vi) Let $u\colon \mathbf{k}_I \to \mathbf{k}_J$ be a morphism of sheaves. As in (i), we have $\mathbf{k}_I(K) \simeq \mathbf{k}$, for any interval $K =]a, b[$, and $\mathbf{k}_J(\mathbb{R}) \simeq 0$. Since u is a morphism of sheaves we have the commutative diagram:

$$\begin{array}{ccc} \mathbf{k} \simeq \mathbf{k}_I(\mathbb{R}) & \xrightarrow{u(\mathbb{R})} & \mathbf{k}_J(\mathbb{R}) = 0 \\ {\scriptstyle r_K^{\mathbb{R}}=\mathrm{id}}\downarrow & & \downarrow \\ \mathbf{k} \simeq \mathbf{k}_I(K) & \xrightarrow{u(K)} & \mathbf{k}_J(K) \end{array}$$

It follows that $u(K) = 0$. This holds for any interval K. Now, any open subset U of $\mathbb{R}$ is a union of disjoint intervals, say $U = \bigcup_{i \in I} K_i$. Then $\mathbf{k}_I(U) \simeq \prod_{i \in I} \mathbf{k}_I(K_i)$, $\mathbf{k}_J(U) \simeq \prod_{i \in I} \mathbf{k}_J(K_i)$ and $u(U) = \prod_{i \in I} u(K_i)$. Hence $u(U) = 0$. Finally $u = 0$.

10. Categories of complexes

Definition 10.1. Let $\mathcal{C}$ be an additive category. A complex $(X^{\cdot}, d_X^{\cdot})$ in $\mathcal{C}$ is a sequence of composable morphisms in $\mathcal{C}$

$$\cdots \to X^i \xrightarrow{d_X^i} X^{i+1} \to \cdots$$

such that $d^{i+1} \circ d^i = 0$, for all $i \in \mathbb{Z}$ (we forget the subscripts when there is no ambiguity). The morphisms d_X^i are called the differentials.

A morphism f from a complex $(X^{\cdot}, d_X^{\cdot})$ to a complex $(Y^{\cdot}, d_Y^{\cdot})$ is a sequence of morphisms $f^i\colon X^i \to Y^i$, $i \in \mathbb{Z}$, commuting with the differentials.

We denote by $\mathbf{C}(\mathcal{C})$ the category of complexes in $\mathcal{C}$. A complex is said bounded from below (resp. above) if $X^i \simeq 0$ for $i \ll 0$ (resp. $i \gg 0$). It is bounded if it is bounded from below and from above. We let $\mathbf{C}^+(\mathcal{C})$, $\mathbf{C}^-(\mathcal{C})$, $\mathbf{C}^b(\mathcal{C})$ be the corresponding categories.

These categories $\mathbf{C}^\bullet(\mathcal{C})$ are additive and the sum (of objects or morphisms) is made "degreewise": $(X \oplus Y)^i = X^i \oplus Y^i, \dots$

We recall that any morphism $f\colon A \to B$ in an abelian category factorizes as $A \xrightarrow{p'_f} \operatorname{im} f \xrightarrow{i'_f} B$ (see (5.1)) and we have the exact sequences (5.2):

$$0 \to \ker f \xrightarrow{i_f} A \xrightarrow{p'_f} \operatorname{im} f \to 0,$$

$$0 \to \operatorname{im} f \xrightarrow{i'_f} B \xrightarrow{p_f} \operatorname{coker} f \to 0.$$

Now let $g\colon B \to C$ be another morphism such that $g \circ f = 0$. Then $(g \circ i'_f) \circ p'_f = 0$ and, since p'_f is an epimorphism, $g \circ i'_f = 0$. We deduce a uniquely defined morphism $j\colon \operatorname{im} f \to \ker g$ such that $i'_f = i_g \circ j$. Since i'_f is a monomorphism, we see that j is also a monomorphism (check!). We write

$$\operatorname{im} f \overset{j}{\hookrightarrow} \ker g \overset{i_g}{\hookrightarrow} B$$

and, since j is a monomorphism, we often use the notation $\operatorname{coker} j = \ker g / \operatorname{im} f$.

Definition 10.2. Let $\mathcal{C}$ be an abelian category and let $X = (X^{\cdot}, d_X^{\cdot}) \in \mathbf{C}(\mathcal{C})$. For $i \in \mathbb{Z}$ we define

$$Z^i(X) = \ker d_X^i, \qquad B^i(X) = \operatorname{im} d_X^{i-1},$$
$$H^i(X) = Z^i(X)/B^i(X) = \operatorname{coker}(B^i(X) \to Z^i(X))$$

and we call $H^i(X)$ the i^{th} *cohomology* of X. (We also call $Z^i(X)$ (resp. $B^i(X)$) the i^{th} *group of cocycles* (resp. *coboundaries*).)

For a morphism of complexes $f\colon X \to Y$ we denote by $Z^i(f)$, $B^i(f)$, $H^i(f)$ the induced morphisms, which exist and are well-defined by Lemma 5.6. We see that Z^i, B^i, H^i are functors from $\mathbf{C}(\mathcal{C})$ to $\mathcal{C}$.

As in Exercise 7.9 we can prove:

Lemma 10.3. *If $\mathcal{C}$ is abelian, then $\mathbf{C}(\mathcal{C})$ is also abelian. Moreover for a morphism $f\colon X \to Y$ in $\mathbf{C}(\mathcal{C})$ the kernel satisfies $(\ker f)^i = \ker(f^i)$ and the differential $d^i_{\ker f}$ is the natural morphism $\ker(f^i) \to \ker(f^{i+1})$ given by Lemma 5.6. The same holds for the cokernel.*

Here are useful lemmas to deal with complexes and long cohomology sequences.

Lemma 10.4. *Let $\mathcal{C}$ be an abelian category. We consider the commutative diagram in $\mathcal{C}$*

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} \\ 0 & \longrightarrow & X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' \end{array} \qquad \textit{resp.} \quad \begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \longrightarrow & 0 \end{array}$$

and we assume that the rows are exact. Then this diagram induces a canonical (in the sense detailed in Proposition 10.7) exact sequence $0 \to \ker u \to \ker v \to \ker w$ (resp. $\operatorname{coker} u \to \operatorname{coker} v \to \operatorname{coker} w \to 0$).

Lemma 10.5 (The snake lemma – see [2] lem. 12.1.1)**.** *Let $\mathcal{C}$ be an abelian category. We consider the commutative diagram in $\mathcal{C}$*

$$\begin{array}{ccccccccc} & & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \\ 0 & \longrightarrow & X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & & \end{array}$$

and we assume that the rows are exact. Then this diagram induces a canonical (in the sense detailed in Proposition 10.7) exact sequence

$$\ker u \to \ker v \to \ker w \to \operatorname{coker} u \to \operatorname{coker} v \to \operatorname{coker} w.$$

Lemma 10.6 (The five lemma)**.** *Let $\mathcal{C}$ be an abelian category. We consider the commutative diagram in $\mathcal{C}$*

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle e} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E'. \end{array}$$

We assume that the rows are exact and that a, b, d, e are isomorphisms. Then c is also an isomorphism.

Proposition 10.7. *Let $\mathcal{C}$ be an abelian category and let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence in $\mathbf{C}(\mathcal{C})$. Then there exists a* canonical *long exact sequence in $\mathcal{C}$*

$$\cdots \to H^n(X) \xrightarrow{H^n(f)} H^n(Y) \xrightarrow{H^n(g)} H^n(Z) \xrightarrow{\delta^n} H^{n+1}(X) \xrightarrow{H^{n+1}(f)} H^{n+1}(Y) \xrightarrow{H^{n+1}(g)} H^{n+1}(Z) \to \cdots .$$

Canonical means here: if we have a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \\ 0 & \longrightarrow & X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \longrightarrow & 0, \end{array}$$

then we obtain a commutative diagram of long exact sequences (in particular the squares
$$\begin{array}{ccc} H^n(Z) & \xrightarrow{\delta^n} & H^{n+1}(X) \\ {\scriptstyle H^n(w)}\downarrow & & \downarrow{\scriptstyle H^{n+1}(u)} \\ H^n(Z') & \xrightarrow{\delta'^n} & H^{n+1}(X') \end{array}$$
).

Proof. For a given i, Lemma 10.4, applied with rows $0 \to X^i \to Y^i \to Z^i$ and $0 \to X^{i+1} \to Y^{i+1} \to Z^{i+1}$ and vertical morphisms $d^i_\bullet$, gives the exact sequence "$E(i)$": $0 \to Z^i(X) \to Z^i(Y) \to Z^i(Z)$. In the same we have the exact sequence "$F(i)$": $\operatorname{coker}(d^i_X) \to \operatorname{coker}(d^i_Y) \to \operatorname{coker}(d^i_Z) \to 0$.

We have morphisms $\operatorname{coker} d^{i-1}_\bullet \to Z^{i+1}_\bullet$ (see Lemma 10.8 below). Then Lemma 10.5, applied with the rows $F(i-1)$ and $E(i+1)$ give the exact sequence of the proposition. □

Lemma 10.8. *Let $\mathcal{C}$ be an abelian category and let $X = (X^\cdot, d^\cdot_X) \in \mathbf{C}(\mathcal{C})$. Then we have the exact sequence*

$$0 \to H^i(X) \to \operatorname{coker}(d^{i-1}_X) \to Z^{i+1}(X) \to H^{i+1}(X) \to 0.$$

Proof. By definition of $H^{i+1}(X)$ we have the exact sequence $X^i \xrightarrow{d^i_X} Z^{i+1}(X) \to H^{i+1}(X) \to 0$. Since $d^i_X \circ d^{i-1}_X = 0$, d^i_X factorizes through

$\operatorname{coker}(d_X^{i-1})$ and gives the end of the sequence in the lemma:

$$\begin{array}{ccccccc}
X^{i-1} & & & & & & \\
\downarrow & & & & & & \\
X^i & \xrightarrow{e} & Z^{i+1}(X) & \longrightarrow & H^{i+1}(X) & \longrightarrow & 0 \\
\downarrow & \overset{f}{\nearrow} & & & & & \\
\operatorname{coker}(d_X^{i-1}) & & & & & &
\end{array}$$

(Note that $\operatorname{im}(f) = \operatorname{im}(e)$ since $X^i \to \operatorname{coker}(d_X^{i-1})$ is an epimorphism.) The beginning of the sequence, $0 \to H^i(X) \to \operatorname{coker}(d_X^{i-1}) \to Z^{i+1}(X)$, follows from the snake lemma 10.5 applied to the diagram

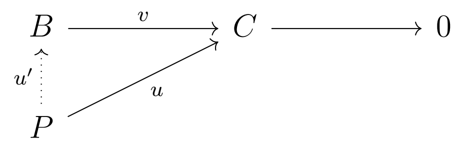
$$\begin{array}{ccccccc}
 & & X^{i-1} & \longrightarrow & X^{i-1} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z^i(X) & \longrightarrow & X^i & \longrightarrow & Z^{i+1}(X). & &
\end{array}$$

□

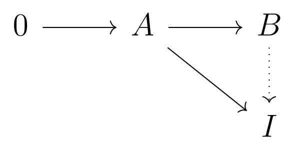
11. Resolutions

In this section $\mathcal{C}$ is often an abelian category but sometimes additive is enough.

Definition 11.1. Let $\mathcal{C}$ be an abelian category. An object $P \in \mathcal{C}$ is *projective* if, for any given epimorphism $v\colon B \to C$ and any $u\colon P \to C$, there exists $u'\colon P \to B$ such that $u = v \circ u'$:



Injective is defined by reversing the arrows:



We say that $\mathcal{C}$ has *enough projectives* if for any $A \in \mathcal{C}$ there exists an epimorphism $P \twoheadrightarrow A$ with P projective. (Enough injectives: $\forall A$, $\exists A \hookrightarrow I$, I injective.)

Definition 11.2. Let $M \in \mathrm{Ob}(\mathcal{C})$. A *left resolution* of M is a complex $X = (X^{\cdot}, d_X^{\cdot}) \in \mathbf{C}(\mathcal{C})$ such that $X^i \simeq 0$ for $i > 0$, together with a morphism $\varepsilon\colon X^0 \to M$ such that the sequence

$$\cdots \to X^{-2} \xrightarrow{d^{-2}} X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{\varepsilon} M \to 0$$

is exact. In particular $H^i(X) \simeq 0$ for all $i \neq 0$ and $H^0(X) \simeq M$. A resolution is called projective if all the X^i's are projective.

A *right resolution* is defined by reversing the arrows (hence we have an exact sequence $0 \to M \xrightarrow{\varepsilon} X^0 \xrightarrow{d^0} \cdots$). It is called injective if all the X^i's are injective.

Proposition 11.3. *Let $\mathcal{C}$ be an abelian category. We assume that $\mathcal{C}$ has enough projectives. Then any $M \in \mathrm{Ob}(\mathcal{C})$ has a projective (left) resolution. Similarly, when we have enough injectives, we have injective right resolutions.*

Proof. By definition of "enough projectives" there exists an epimorphism $P^0 \xrightarrow{\varepsilon} M$ with P^0 projective. We set $M^1 = \ker \varepsilon \xrightarrow{i^1} P^0$ and choose an epimorphism $P^{-1} \xrightarrow{e^1} M^1$. We set $d^{-1} = i^1 \circ e^1$. We then have the exact sequence $P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{\varepsilon} M \to 0$. We set $M^2 = \ker d^{-1}$ and proceed with M^2 as with M^1. We go on and obtain the resolution by induction. $\square$

The next proposition says that a projective resolution is unique up to homotopy in the following sense.

Definition 11.4. Let $\mathcal{C}$ be an additive category and let $P = (P^{\cdot}, d_P^{\cdot})$, $Q = (Q^{\cdot}, d_Q^{\cdot}) \in \mathbf{C}(\mathcal{C})$. We say that two morphisms $f, g\colon P \to Q$ in $\mathbf{C}(\mathcal{C})$ are homotopic if there exists a family of morphisms $s^i\colon P^i \to Q^{i-1}$, $i \in \mathbb{Z}$, such that

$$f^n - g^n = d_Q^{n-1} \circ s^n + s^{n+1} \circ d_P^n,$$

for all $n \in \mathbb{Z}$.

Proposition 11.5. *Let $\mathcal{C}$ be an abelian category, let $M \in \mathrm{Ob}(\mathcal{C})$ and let $P = (P^{\cdot}, d_P^{\cdot}) \in \mathbf{C}(\mathcal{C})$ together with $\varepsilon\colon P^0 \to M$ be a projective resolution of M. Let $f'\colon M \to N$ be a morphism in $\mathcal{C}$. Let $X = (X^{\cdot}, d_X^{\cdot}) \in \mathbf{C}(\mathcal{C})$ together with $\eta\colon X^0 \to N$ be a left resolution of N. Then there exists a morphism $f\colon P \to X$ in $\mathbf{C}(\mathcal{C})$ lifting f' in the sense that $f' \circ \varepsilon = \eta \circ f^0$. In other words there exists a commutative diagram*

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & P^{-2} & \xrightarrow{d_P^{-2}} & P^{-1} & \xrightarrow{d_P^{-1}} & P^0 & \xrightarrow{\varepsilon} & M \\
 & \downarrow{\scriptstyle f^{-2}} & & \downarrow{\scriptstyle f^{-1}} & & \downarrow{\scriptstyle f^{0}} & & \downarrow{\scriptstyle f'} \\
\cdots \longrightarrow & X^{-2} & \xrightarrow[d_X^{-2}]{} & X^{-1} & \xrightarrow[d_X^{-1}]{} & X^0 & \xrightarrow[\eta]{} & N.
\end{array}$$

Moreover, if $g\colon P \to X$ is another morphism lifting f', then f and g are homotopic.

Proof. (i) The existence of f^0 follows from the facts that η is an epimorphism and P^0 is projective. Then we remark that $f^0 \circ d_P^{-1}$ factorizes through $\ker \eta = \operatorname{im} d_X^{-1}$. Hence we obtain f^{-1} in the same way, using the facts that d_X^{-1} is an epimorphism to its image and P^{-1} is projective. We obtain all f^k in this way inductively.

(ii) We set $h^k = f^k - g^k$. We have the commutative diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & P^{-2} & \xrightarrow{d_P^{-2}} & P^{-1} & \xrightarrow{d_P^{-1}} & P^0 & \xrightarrow{\varepsilon} & M \\
 & \downarrow{\scriptstyle h^{-2}} & & \downarrow{\scriptstyle h^{-1}} & & \downarrow{\scriptstyle h^{0}} & & \downarrow{\scriptstyle 0} \\
\cdots \longrightarrow & X^{-2} & \xrightarrow[d_X^{-2}]{} & X^{-1} & \xrightarrow[d_X^{-1}]{} & X^0 & \xrightarrow[\eta]{} & N.
\end{array}$$

Since $\eta \circ h^0 = 0$, h^0 factorizes through $i^0\colon P^0 \to \ker \eta = \operatorname{im} d_X^{-1}$. Since $X^{-1} \to \operatorname{im} d_X^{-1}$ and P^0 is projective, we can lift i^0 to $s^0\colon P^0 \to X^{-1}$. We then have $h^0 = d_X^{-1} \circ s^0$.

We define $h'^{-1} = h^{-1} - s^0 \circ d_P^{-1}$. Then $d_X^{-1} \circ h'^{-1} = 0$ and we can apply the same procedure to find $s^{-1}\colon P^{-1} \to X^{-2}$ such that $h'^{-1} = d_X^{-2} \circ s^{-1}$. Hence $h^{-1} = s^0 \circ d_P^{-1} + d_X^{-2} \circ s^{-1}$. Now we go on inductively. □

We obtain the injective versions of Propositions 11.3 and 11.5 by reversing the arrows.

The homotopy relation is compatible with the additive structure of $\mathrm{Hom}(P, Q)$ and with the composition in $\mathbf{C}(\mathcal{C})$. It follows that we can define a category of *complexes up to homotopy* as follows.

Definition 11.6. Let $\mathcal{C}$ be an additive category. We define a category $\mathbf{K}(\mathcal{C})$ by $\mathrm{Ob}(\mathbf{K}(\mathcal{C})) = \mathrm{Ob}(\mathbf{C}(\mathcal{C}))$ and

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{C})}(P, Q) = \mathrm{Hom}_{\mathbf{C}(\mathcal{C})}(P, Q)/\sim_h,$$

where $\sim_h$ is the homotopy relation on $\mathrm{Hom}_{\mathbf{C}(\mathcal{C})}(P, Q)$. Then $\mathbf{K}(\mathcal{C})$ is an additive category.

We have an obvious functor $\mathbf{C}(\mathcal{C}) \to \mathbf{K}(\mathcal{C})$ which is the identity on objects and the quotient map on the morphisms.

It is easy to check that two homotopic morphisms of complexes induce the same morphism on homology (when $\mathcal{C}$ is abelian):

Lemma 11.7. *Let $\mathcal{C}$ be an abelian category and let $f\colon X \to Y$ be a morphism in $\mathbf{C}(\mathcal{C})$. We assume that f is homotopic to the zero morphism. Then $H^i(f)\colon H^i(X) \to H^i(Y)$ is zero, for all $i \in \mathbb{Z}$. In particular the homology functors $H^i\colon \mathbf{C}(\mathcal{C}) \to \mathcal{C}$ induce well-defined functors $H^i\colon \mathbf{K}(\mathcal{C}) \to \mathcal{C}$.*

Let $\mathbf{K}_{pr}(\mathcal{C})$ be the full subcategory of $\mathbf{K}(\mathcal{C})$ formed by the complexes $P = (P^{\cdot}, d_P^{\cdot})$ such that $P^i = 0$ for $i > 0$, $H^i(P) \simeq 0$ for all $i \neq 0$ and P^i is projective for each $i \leq 0$ ("*pr*" stands for "projective resolution"). Then Propositions 11.3 and 11.5 give

Proposition 11.8. *The functor $H^0\colon \mathbf{K}_{pr}(\mathcal{C}) \to \mathcal{C}$ is essentially surjective and fully faithful. In other words, it is an equivalence and we can find a quasi-inverse $\mathbf{res}_{pr}\colon \mathcal{C} \to \mathbf{K}_{pr}(\mathcal{C})$.*

Different choices of inverse to $H^0\colon \mathbf{K}_{pr}(\mathcal{C}) \to \mathcal{C}$ give different functors $\mathbf{res}_{pr}$ but they are (canonically) isomorphic by the following remark.

Remark 11.9. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between two categories which is an equivalence of categories. Let $G_1, G_2\colon \mathcal{B} \to \mathcal{A}$ be inverses of F, together with isomorphisms of functors $\varepsilon_i\colon F \circ G_i \xrightarrow{\sim} \mathrm{id}_{\mathcal{B}}$. Then there exists a unique isomorphism of functors $\varepsilon\colon G_1 \xrightarrow{\sim} G_2$ such that $F \circ \varepsilon = \varepsilon_2 \circ \varepsilon_1^{-1}$, where $F \circ \varepsilon$ denotes abusively the morphism given by $(F \circ \varepsilon)(X) = F(\varepsilon(X))$, for $X \in \mathcal{A}$.

Indeed, we must define $\varepsilon(Y)\colon G_1(Y) \to G_2(Y)$, $Y \in \mathcal{B}$, as the inverse image of $\varepsilon_2(Y) \circ \varepsilon_1^{-1}(Y)$ by the bijection $\mathrm{Hom}(G_1(Y), G_2(Y)) \xrightarrow{\sim} \mathrm{Hom}(F \circ G_1(Y), F \circ G_2(Y))$, and we can check that this gives an isomorphism of functors.

Definition 11.10. A functor $F\colon \mathcal{C} \to \mathcal{C}'$ between additive categories is *additive* if F commutes with $\oplus$ and, for all $A, B \in \mathcal{C}$, the maps $\mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$, $f \mapsto F(f)$, are group morphisms.

An additive functor $F\colon \mathcal{C} \to \mathcal{C}'$ between abelian categories is *exact* if it sends short exact sequences to short exact sequences. It is *left exact* if, for any exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C$, the sequence $0 \to F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$ is exact. It is *right exact* if, for any exact sequence $A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the sequence $F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \to 0$ is exact.

Definition 11.11. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be an additive functor between additive categories. We define $\mathbf{C}(F)\colon \mathbf{C}(\mathcal{C}) \to \mathbf{C}(\mathcal{C}')$ by $F(X^\cdot, d_X^\cdot) = (F(X^\cdot), F(d_X^\cdot))$. Then $\mathbf{C}(F)$ is additive and compatible with homotopy. It induces an additive functor $\mathbf{K}(F)\colon \mathbf{K}(\mathcal{C}) \to \mathbf{K}(\mathcal{C}')$.

Definition 11.12. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a right exact functor between two abelian categories. We assume that $\mathcal{C}$ has enough projectives. For $i \in \mathbb{Z}$ we define a functor $L^iF\colon \mathcal{C} \to \mathcal{C}'$ by $L^iF = H^i \circ \mathbf{K}(F) \circ \mathbf{res}_{pr}$, where $\mathbf{res}_{pr}\colon \mathcal{C} \to \mathbf{K}_{pr}(\mathcal{C})$ is given by Proposition 11.8.

As already remarked after Proposition 11.8 (see Remark 11.9) two choices of $\mathbf{res}_{pr}$ give isomorphic functors. In particular, for any $M \in \mathcal{C}$ and $i \in \mathbb{Z}$, we have $L^iF(M) = H^i(\mathbf{K}(F)(P))$, where P is any projective resolution of M.

The right exactness of F ensures that

$$L^0F(M) \simeq F(M), \qquad \text{for all } M \in \mathrm{Ob}(\mathcal{C}).$$

Lemma 11.13. *Let $\mathcal{C}$ be an additive category and $P, R \in \mathcal{C}$. The natural morphisms $i = \binom{\mathrm{id}_P}{0}\colon P \to P \oplus R$ and $p = (0, \mathrm{id}_R)\colon P \oplus R \to R$ give an exact sequence $0 \to P \to P \oplus R \to R \to 0$ (that is, i is a monomorphism and it has a cokernel which is p).*

Proposition 11.14. *Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence in $\mathcal{C}$. Let $P = (P^\cdot, d_P^\cdot) \in \mathbf{C}(\mathcal{C})$ together with $\varepsilon\colon P^0 \to X$ be a projective resolution of X. Let $R = (R^\cdot, d_R^\cdot) \in \mathbf{C}(\mathcal{C})$ together with $\gamma\colon R^0 \to Z$ be a projective resolution of Z. We set $Q^k = P^k \oplus R^k$. Then we can find a differential $d_Q^\cdot$ and $\eta\colon Q^0 \to Y$ turning $(Q^\cdot, d_Q^\cdot)$ into a projective resolution of Y such that the natural morphisms $i_k\colon P^k \to$*

Q^k, $p_k\colon Q^k \to R^k$ give a commutative diagram of resolutions:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & P^{-2} & \xrightarrow{d_P^{-2}} & P^{-1} & \xrightarrow{d_P^{-1}} & P^0 & \xrightarrow{\varepsilon} & X \\
 & & \downarrow{\scriptstyle i^{-2}} & & \downarrow{\scriptstyle i^{-1}} & & \downarrow{\scriptstyle i^{0}} & & \downarrow{\scriptstyle f} \\
\cdots & \dashrightarrow & Q^{-2} & \xrightarrow{d_Q^{-2}} & Q^{-1} & \xrightarrow{d_Q^{-1}} & Q^0 & \xrightarrow{\gamma} & Y \\
 & & \downarrow{\scriptstyle p^{-2}} & & \downarrow{\scriptstyle p^{-1}} & & \downarrow{\scriptstyle p^{0}} & & \downarrow{\scriptstyle g} \\
\cdots & \longrightarrow & R^{-2} & \xrightarrow[d_R^{-2}]{} & R^{-1} & \xrightarrow[d_R^{-1}]{} & R^0 & \xrightarrow[\eta]{} & Z.
\end{array}$$

Proof. Since g is an epimorphism and R^0 is projective, we can factorize g through $\eta'\colon R^0 \to Y$. Then $\gamma = (f \circ \varepsilon, \eta')\colon Q^0 = P^0 \oplus R^0 \to Y$ gives the commutative squares. Moreover γ is an epimorphism: if $a\colon Y \to M$ is such that $a \circ \gamma = 0$, then $a \circ f \circ \varepsilon = 0$, hence $a \circ f = 0$ (because ε is an epimorphism), hence a factorizes through g by $a'\colon Z \to M$; we then obtain $a' \circ \eta = 0$, hence $a' = 0$ (since η is an epimorphism), hence $a = 0$.

The snake lemma gives the exact sequence $0 \to \ker \varepsilon \to \ker \gamma \to \ker \eta \to 0$. We replace the initial exact sequence by this one and P^0, Q^0, R^0 by P^{-1}, Q^{-1}, R^{-1}. The same argument gives an epimorphism $e^{-1}\colon Q^{-1} \to \ker \gamma$ making commutative squares. We let d_Q^{-1} be the composition of e^{-1} and the morphism $\ker \gamma \to Q^0$.

Now we go on by induction. □

Theorem 11.15. *Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a right exact functor between two abelian categories. We assume that $\mathcal{C}$ has enough projectives. Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence in $\mathcal{C}$. Then there exists a canonical long exact sequence in $\mathcal{C}'$*

$$\cdots \to L^nF(X) \xrightarrow{L^nF(f)} L^nF(Y) \xrightarrow{L^nF(g)} L^nF(Z) \xrightarrow{\delta^n} L^{n+1}F(X)$$
$$\xrightarrow{L^{n+1}F(f)} L^{n+1}F(Y) \xrightarrow{L^{n+1}F(g)} L^{n+1}F(Z) \to \cdots .$$

More precisely, if we have a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \\
0 & \longrightarrow & X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \longrightarrow & 0,
\end{array}$$

then all squares

$$\begin{array}{ccc} L^nF(Z) & \xrightarrow{\delta^n} & L^{n+1}F(X) \\ \downarrow{\scriptstyle L^nF(w)} & & \downarrow{\scriptstyle L^nF(u)} \\ L^nF(Z') & \xrightarrow{\delta'^n} & L^{n+1}F(X') \end{array}$$

commute.

Proof. We use the result and the notations of Proposition 11.14. By Lemma 11.13 the sequence of complexes $0 \to P^\cdot \to Q^\cdot \to R^\cdot \to 0$ is exact. We apply the functor $C(F)$ to the sequence and obtain a sequence in $C(\mathcal{C}')$: $0 \to F(P^\cdot) \to F(Q^\cdot) \to F(R^\cdot) \to 0$. Since F is additive, we have $F(Q^i) \simeq F(P^i) \oplus F(R^i)$ and the morphisms still are the natural morphisms from/to a product/sum. Hence this sequence in $C(\mathcal{C}')$ is also exact, by Lemma 11.13 again. Now the theorem follows from Proposition 10.7. □

Actually we only used the additivity of F in the proof of the theorem. The hypothesis that F is right exact is needed to make the connection between F and L^0F:

Lemma 11.16. *With the hypothesis of Theorem 11.15 we have $L^0F \simeq F$.*

Proof. Let $X \in \mathcal{C}$ be given with a projective resolution $P^\cdot$ together with $\varepsilon\colon P^0 \to X$ such that $\cdots \to P^{-1} \to P^0 \to X \to 0$ is exact. The hypothesis says that $F(P^{-1}) \to F(P^0) \to F(X) \to 0$ is exact, which means $F(X) \simeq \operatorname{coker}(F(d_P^{-1}))$. On the other hand $H^0(\cdots \to F(P^{-1}) \to F(P^0) \to 0) \simeq \operatorname{coker}(F(d_P^{-1}))$ by definition, which gives the result. □

Definition 11.12, Proposition 11.8 and Theorem 11.15 have analogs for left exact functors in the case where $\mathcal{C}$ has enough injectives. In particular we can define $\mathbf{K}_{ir}(\mathcal{C})$ to be the full subcategory of $\mathbf{K}(\mathcal{C})$ formed by the complexes $I = (I^\cdot, d_I)$ such that $I^i = 0$ for $i < 0$, $H^i(I) \simeq 0$ for all $i \neq 0$ and I^i is injective for each $i \geq 0$ ("ir" stands for "injective resolution"). Then $H^0\colon \mathbf{K}_{ir}(\mathcal{C}) \to \mathcal{C}$ is essentially surjective and fully faithful as in Proposition 11.8 and we can find a quasi-inverse $\mathbf{res}_{ir}\colon \mathcal{C} \to \mathbf{K}_{ir}(\mathcal{C})$. If $F\colon \mathcal{C} \to \mathcal{C}'$ is a left exact functor, we define $R^iF(M) = H^i(\mathbf{K}(F)(\mathbf{res}_{ir}(M)))$. Since F is left exact, we can see that $R^0F = F$. We then have an analog of Theorem 11.15 by replacing all L^n by R^n.

Example 11.17. Let G be a group and $F = (-)_G\colon G - \mathrm{Mod} \to \mathbf{Ab}$ the functor of coinvariants. We have seen that it is right exact. Let us compute $L^iF(\mathbb{Z})$, where $\mathbb{Z}$ is the trivial representation, when $G = \mathbb{Z}/n\mathbb{Z}$ is a finite cyclic group.

We first define the group ring of G, for any group G. Let $\mathbb{Z}[G]$ be the free abelian group generated by the set G, which means $\mathbb{Z}[G] = \mathbb{Z}^{(G)}$, or, $\mathbb{Z}[G] = \{\sum_{g\in G} n_g e_g\}$, where $\{e_g\}$ is the canonical base (if there is no ambiguity, we may even write g instead of e_g) and the n_g in the sum are all 0 but a finite number of them. Then $\mathbb{Z}[G]$ is an abelian group for the termwise sum and also a ring, where the multiplication is induced by the relation $e_g e_h = e_{gh}$. In other words $(\sum_{g\in G} n_g e_g)(\sum_{g\in G} n'_g e_g) = \sum_{g\in G}(\sum_{h\in G} n_h n'_{h^{-1}g})e_g$.

Now $\mathbb{Z}[G]$ is also a G-module for the action $g\cdot x = e_g x$ (it is called the *regular* representation of G). We can see that, for any $M \in G-\mathrm{Mod}$, the map

$$\mathrm{Hom}_{G-\mathrm{Mod}}(\mathbb{Z}[G], M) \to M \qquad u \mapsto u(e_{1_G}),$$

is an isomorphism of abelian groups. It follows easily that $\mathbb{Z}[G]$ is projective.

Now we assume $G = \mathbb{Z}/n\mathbb{Z}$ and define $N = \sum_g e_g \in \mathbb{Z}[G]$ (the *norm* element) and $\delta = e_0 - e_1$. We see that $e_g N = N e_g = N$ for any g, hence $N\delta = \delta N = 0$. We even have the exact sequence

$$\cdots \xrightarrow{\cdot\delta} \mathbb{Z}[G] \xrightarrow{\cdot N} \mathbb{Z}[G] \xrightarrow{\cdot\delta} \mathbb{Z}[G] \xrightarrow{\cdot N} \mathbb{Z}[G] \xrightarrow{\cdot\delta} \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

where $\varepsilon(\sum_{g\in G} n_g e_g) = \sum n_g$, and $\cdot N$, $\cdot\delta$ are the multiplication on the right (which are morphisms of left modules).

Now $(\mathbb{Z}[G])_G \simeq \mathbb{Z}$ is the quotient of $\mathbb{Z}[G]$ by $\mathrm{im}(\cdot\delta)$. It follows that

$$H_i(G,\mathbb{Z}) \simeq H^{-i}\Big(\cdots \xrightarrow{\bar\delta} \mathbb{Z} \xrightarrow{\bar N} \mathbb{Z} \xrightarrow{\bar\delta} \mathbb{Z} \xrightarrow{\bar N} \mathbb{Z} \xrightarrow{\bar\delta} \mathbb{Z}),$$

where $\bar\delta$ and $\bar N$ are the maps induced by $\cdot\delta$ and $\cdot N$. We see that $\bar\delta = 0$ and $\bar N$ is the multiplication by n. Hence

$$H_i(G,\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ \mathbb{Z}/n\mathbb{Z} & \text{if } i = 1,3,5,\dots, \\ 0 & \text{if } i = 2,4,6,\dots. \end{cases}$$

Remark 11.18. In Example 11.17 to see that $\mathbb{Z}[G]$ we can use the more general fact that in the category of modules over a ring R, the free module $R^{(I)}$ is projective, for any family I, and that $G-\mathrm{Mod}$ is equivalent to $\mathrm{Mod}(\mathbb{Z}[G])$.

Indeed a structure of G-module on an abelian group A extends by linearity as a structure of left module over $\mathbb{Z}[G]$, by setting $(\sum n_g e_g)x := \sum n_g(g\cdot x)$, for $x \in A$. The converse is easy: a $\mathbb{Z}[G]$-module structure on A gives a G-module structure by $g\cdot x := e_g x$.

Now, to see that $R^{(I)}$ is projective in $\mathrm{Mod}(R)$, we use

$$\mathrm{Hom}_{\mathrm{Mod}(R)}(R^{(I)}, M) \simeq M^I,$$

for any $M \in \mathrm{Mod}(R)$.

Lemma 11.19. *Let $\mathcal{C}$ be an abelian category. Let A be a (maybe infinite) set.*

Let P_α, $\alpha \in A$, be projective objects. We assume that $P = \bigoplus_{\alpha\in A} P_\alpha$ exists. Then P is projective.

Let I_α, $\alpha \in A$, be injective objects. We assume that $I = \prod_{\alpha\in A} I_\alpha$ exists. Then I is injective.

Proof. This follows from $\mathrm{Hom}_{\mathcal{C}}(\bigoplus_{\alpha\in A} P_\alpha, X) = \prod_{\alpha\in A} \mathrm{Hom}_{\mathcal{C}}(P_\alpha, X)$ and $\mathrm{Hom}_{\mathcal{C}}(X, \prod_{\alpha\in A} I_\alpha) = \prod_{\alpha\in A} \mathrm{Hom}_{\mathcal{C}}(X, I_\alpha)$. □

Lemma 11.20. *An abelian group is injective if and only if it is* divisible *that is, for any $r \neq 0 \in \mathbb{Z}$ and $a \in A$, there exists $b \in A$ such that $a = rb$.*

Sketch of proof. We assume A is divisible and consider an inclusion of abelian groups $M \subset N$ and a morphism $f\colon M \to A$. We choose $x \in N \setminus M$ and prove that f can be extended to the subgroup $M + \langle x\rangle$ of N; then we could conclude by Zorn Lemma (left to the reader). We have the exact sequence $0 \to M \cap \langle x\rangle \to M \oplus \langle x\rangle \to M + \langle x\rangle \to 0$. We have $\langle x\rangle \simeq \mathbb{Z}/n\mathbb{Z}$ (we assume $n \neq 0$ – the other case is similar) and $M \cap \langle x\rangle \simeq \mathbb{Z}/m\mathbb{Z}$ for some $m \mid n$. Then $y = \frac{n}{m}x$ is a generator of $M \cap \langle x\rangle$.

By hypothesis there exists $a \in A$ such that $f(y) = \frac{n}{m}a$. We remark that $na = mf(y) = f(my) = f(0) = 0$; hence we can define $f'\colon M \oplus \langle x\rangle \to A$, $(z, kx) \mapsto f(z) - ka$. Then $f'|_{M\cap\langle x\rangle} = 0$ and f' factorizes through $f''\colon M + \langle x\rangle \to A$ which is the required extension of f. □

Lemma 11.21. *The category* **Ab** *has enough projectives and enough injectives.*

Proof. (i) We know that $\mathbb{Z}$ is projective. For $M \in$ **Ab** and $x \in M$ we define $\varphi_x\colon \mathbb{Z} \to M$, $n \mapsto n \cdot x$. Now the sum of all these maps $\mathbb{Z}^{(M)} \to M$, $(n_x)_{x\in M} \mapsto \sum n_x \cdot x$ is surjective (note that the sum is finite). By Lemma 11.19 $\mathbb{Z}^{(M)}$ is projective.

(ii) Let $M \in$ **Ab**. For any $x \neq 0 \in M$ we can find a morphism $\psi_x\colon M \to \mathbb{Q}/\mathbb{Z}$ such that $\psi_x(x) \neq 0$. Indeed we first define ψ_x on the subgroup $\langle x\rangle \subset M$ by $\psi_x(nx) = [n/m]$, where m is the order of x, if $m \neq \infty$, and $\psi_x(nx) = [n/2]$ if $m = \infty$; then we can extend ψ_x to M since $\mathbb{Q}/\mathbb{Z}$ is injective (by Lemma 11.20). Now we make the product of these maps and define $\psi\colon M \to \mathbb{Q}/\mathbb{Z}^M$, $y \mapsto (\psi_x(y))_{x\in M}$. Then ψ is injective and $\mathbb{Q}/\mathbb{Z}^M$ is injective. □

12. Partial Exam

Exercise 12.1. Let $\mathcal{C}$ be an abelian category and let $\mathrm{Mor}(\mathcal{C})$ be the category of morphisms in $\mathcal{C}$: its objects are the morphisms of $\mathcal{C}$, $(X \xrightarrow{f} Y)$, and its morphisms are the commutative diagrams, $\mathrm{Hom}_{\mathrm{Mor}(\mathcal{C})}((X \xrightarrow{f} Y), (X' \xrightarrow{f'} Y')) = \{(u, v);$ $\begin{array}{ccc} X & \xrightarrow{u} & X' \\ \downarrow f & & \downarrow f' \\ Y & \xrightarrow{v} & Y' \end{array}$ commutes$\}$.

We admit that $\mathrm{Mor}(\mathcal{C})$ is abelian with sums, kernels, cokernels given termwise (for example with the above notations $\ker(u, v) = (\ker(u) \xrightarrow{\bar{f}} \ker(v))$ where $\bar{f}$ is induced by f).

We admit also that, if $I \in \mathcal{C}$ is injective, then $(I \xrightarrow{\mathrm{id}_I} I)$ and $(I \to 0)$ are injective in $\mathrm{Mor}(\mathcal{C})$.

(i) We define $K\colon \mathrm{Mor}(\mathcal{C}) \to \mathcal{C}$, $(X \xrightarrow{u} Y) \mapsto \ker(u)$. Prove that K is a left exact functor. Give an example showing that it is not right exact.

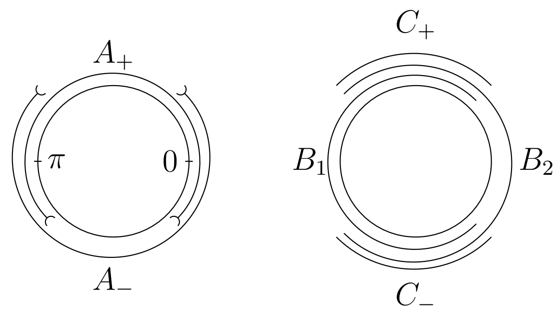
(ii) In the case $\mathcal{C} = \mathbf{Vect}(\mathbf{k})$, the category of vector spaces over some field (so any object is injective), give a right injective resolution of a general object $(X \xrightarrow{u} Y)$ and compute the derived functors R^iK.

Exercise 12.2. Let $\mathbf{k}$ be a field. We recall the notation $\mathbf{k}_Z$, for a topological space X and a closed subset $Z \subset X$: $\mathbf{k}_Z$ is the sheaf defined by $\mathbf{k}_Z(U) = \{f\colon Z \cap U \to \mathbf{k};\ f \text{ is locally constant}\}$, for $U \in \mathrm{Op}(X)$. For $Z' \subset Z$ another closed subset, we have a natural morphism $i\colon \mathbf{k}_Z \to \mathbf{k}_{Z'}$ given by the morphisms $i(U)\colon \mathbf{k}_Z(U) \to \mathbf{k}_{Z'}(U)$ such that $i(U)(f) = f|_{U \cap Z'}$.

(i) Let $a < b < c < d \in \mathbb{R}$. Let $I_1, I_2, J \subset \mathbb{R}$ be the closed intervals $I_1 = [a, c]$, $I_2 = [b, d]$, $J = I_1 \cap I_2 = [b, c]$. We let $i_1\colon \mathbf{k}_{I_1} \to \mathbf{k}_J$, $i_2\colon \mathbf{k}_{I_2} \to \mathbf{k}_J$ be the above natural morphisms. For $\alpha \in \mathbf{k}$, $\alpha \neq 0$, we define $u_\alpha\colon \mathbf{k}_{I_1} \oplus \mathbf{k}_{I_2} \to \mathbf{k}_J$ by $u_\alpha = (i_1, \alpha i_2)$ (in other words $u_\alpha(U)(f_1, f_2) = f_1|_{U \cap J} + \alpha f_2|_{U \cap J}$).

• Prove that $\ker(u_\alpha) \simeq \mathbf{k}_I$, where $I = [a, d]$. More precisely, let $j_1\colon \mathbf{k}_I \to \mathbf{k}_{I_1}$, $j_2\colon \mathbf{k}_I \to \mathbf{k}_{I_2}$ be the natural morphisms; prove that $v = (\alpha j_1, -j_2)\colon \mathbf{k}_I \to \mathbf{k}_{I_1} \oplus \mathbf{k}_{I_2}$ induces $\mathbf{k}_I \xrightarrow{\sim} \ker(u_\alpha)$.

(ii) Let $\theta = \pi/4$. On the circle S^1 we consider the two arcs $A_+ =]-\theta, \pi + \theta[$ (the upper arc containing $\pi/2$) and $A_- =]\pi - \theta, \theta[$ (the lower arc containing $-\pi/2$).



Let F be a sheaf on S^1. We denote by $F|_{A_+}$ the sheaf restricted to A_+, that is, $F|_{A_+}(U) = F(U)$ for $U \in \mathrm{Op}(A_+)$ (and similarly for $F|_{A_-}$). We assume that

$$F|_{A_+} \simeq \mathbf{k}_{A_+} \text{ and } F|_{A_-} \simeq \mathbf{k}_{A_-}. \tag{12.1}$$

In particular $\Gamma(A_+; F) \simeq \Gamma(A_+; \mathbf{k}_{A_+}) \simeq \mathbf{k}$ and $\Gamma(A_-; F) \simeq \mathbf{k}$. We also have $F_\pi \simeq \mathbf{k}$, $F_0 \simeq \mathbf{k}$. Let $s_+ \neq 0 \in \Gamma(A_+; F)$ and $s_- \neq 0 \in \Gamma(A_-; F)$. Since F_π and F_0 are of dimension 1, there exist $c_\pi, c_0 \neq 0 \in \mathbf{k}$ such that

$$(s_-)_\pi = c_\pi \cdot (s_+)_\pi, \qquad (s_-)_0 = c_0 \cdot (s_+)_0.$$

• Prove that $m(F) = c_\pi / c_0$ is independent of the choice of s_+, s_-. We call it the monodromy of F.

• Let G be another sheaf on S^1 satisfying (12.1). We assume that there exists an isomorphism $f\colon F \xrightarrow{\sim} G$. Prove that $m(F) = m(G)$.

Extra question

(iii) Let $B_1 = [\theta, -\theta]$ be the left arc (containing π) and $B_2 = [\pi+\theta, \pi-\theta]$ be the right arc (containing 0), $C_+ = [\theta, \pi-\theta]$, $C_- = [\pi+\theta, -\theta]$. Let $i_{1+}\colon \mathbf{k}_{B_1} \to \mathbf{k}_{C_+}$ be the natural morphism and similarly i_{1-}, i_{2+}, i_{2-}. For $\alpha \neq 0 \in \mathbf{k}$ we define

$$F_\alpha = \ker(f_\alpha\colon \mathbf{k}_{B_1} \oplus \mathbf{k}_{B_2} \to \mathbf{k}_{C_+} \oplus \mathbf{k}_{C_-})$$

where $f_\alpha = \begin{pmatrix} i_{1+} & i_{2+} \\ i_{1-} & \alpha i_{2-} \end{pmatrix}$. We know by (i) that F_α satisfies (12.1). What is the monodromy of F_α?

13. Correction of the partial exam

Correction of Exercise 12.1. The zero object in $\mathrm{Mor}(\mathcal{C})$ is $(0 \to 0)$. A short sequence in $\mathrm{Mor}(\mathcal{C})$, say $(0 \to 0) \to (X \xrightarrow{f} Y) \xrightarrow{(u,v)} (X' \xrightarrow{f'} Y') \xrightarrow{(u',v')} (X'' \xrightarrow{f''} Y'') \to (0 \to 0)$, is given by a commutative diagram;

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{u} & X' & \xrightarrow{u'} & X'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f''} & & \\
0 & \longrightarrow & Y & \xrightarrow{v} & Y' & \xrightarrow{v'} & Y'' & \longrightarrow & 0
\end{array}$$

Since ker and coker in $\mathrm{Mor}(\mathcal{C})$ are computed "objectwise", this short sequence is exact if and only if both rows are exact.

(i) By Lemma 10.4 the above exact sequence induces an exact sequence of kernels $0 \to \ker(f) \to \ker(f') \to \ker(f'')$. Hence the functor K, $(X \xrightarrow{f} Y) \mapsto \ker(f)$, is left exact. The snake lemma even says that this sequence goes on as $\cdots \to \ker(f'') \to \mathrm{coker}(f) \to \cdots$. Hence to find an example were the sequence of kernels is not right exact, we have to try examples where f is not an epimorphism. The easiest case seems to be

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & Y & \xrightarrow{\mathrm{id}_Y} & Y & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \mathrm{id}_Y} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & Y & \xrightarrow{\mathrm{id}_Y} & Y & \xrightarrow{0} & 0 & \longrightarrow & 0
\end{array}$$

which gives the sequence of kernels $0 \to 0 \xrightarrow{\bar{u}} 0 \xrightarrow{\bar{u}'} Y$, where indeed $\bar{u}'$ is not an epimorphism (for $Y \neq 0$).

(ii) We first find a monomorphism from $(X \xrightarrow{f} Y)$ to an injective $(I_0 \xrightarrow{u_0} J_0)$, compute the cokernel and go on by induction.

We know that $(X \xrightarrow{\mathrm{id}_X} X)$, $(Y \xrightarrow{\mathrm{id}_Y} Y)$, $(X \to 0)$, $(Y \to 0)$ are injective. We have no obvious map $Y \to X$, so the natural maps we can consider are

$$\begin{array}{ccc}
X & \xrightarrow{\mathrm{id}_X} & X \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle 0} \\
Y & \xrightarrow{0} & 0
\end{array}
\qquad
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle 0} \\
Y & \xrightarrow{0} & 0
\end{array}
\qquad
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}_Y} \\
Y & \xrightarrow{\mathrm{id}_Y} & Y
\end{array}$$

The sum of the first and the third morphisms gives a monomorphism

$$\begin{array}{ccc}
X & \xrightarrow{u} & X \oplus Y \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle (0,\mathrm{id}_Y)} \\
Y & \xrightarrow{\mathrm{id}_Y} & Y
\end{array}
\qquad \text{where } u = \begin{pmatrix} \mathrm{id}_X \\ f \end{pmatrix}.$$

The cokernel of this morphism is $(Y \to 0)$ which is already injective. More precisely we have the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{\binom{\mathrm{id}_X}{f}} & X \oplus Y & \xrightarrow{(f,-\mathrm{id}_Y)} & Y & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow (0,\mathrm{id}_Y) & & \downarrow & & \\ 0 & \longrightarrow & Y & \xrightarrow{\mathrm{id}_Y} & Y & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

So the process stops here and our resolution is $\begin{array}{ccc} X \oplus Y & \xrightarrow{(f,-\mathrm{id}_Y)} & Y \\ \downarrow (0,\mathrm{id}_Y) & & \downarrow \\ Y & \longrightarrow & 0 \end{array}$

Applying the functor K to the resolution we find the complex $C = \cdots \to 0 \to X \xrightarrow{f} Y \to 0 \to \cdots$, with X in degree 0. The cohomology of this complex is $H^0(C) = \ker(f)$, $H^1(C) = \mathrm{coker}(f)$ and 0 in degree $\neq 0, 1$. In other words, $R^0K = K$, $R^1K = \mathrm{coker}$, $R^iK = 0$, $i \neq 0, 1$.

Correction of Exercise 12.2. (i) We consider the short sequence

$$0 \to \mathbf{k}_I \xrightarrow{\binom{\alpha j_1}{-j_2}} \mathbf{k}_{I_1} \oplus \mathbf{k}_{I_2} \xrightarrow{(i_1, \alpha i_2)} \mathbf{k}_J$$

We set $u_\alpha = (i_1, \alpha i_2)$, $v = \binom{\alpha j_1}{-j_2}$. A sequence of sheaves is exact if and only if the sequences of stalks are exact at each point x. There are four possibilities

- $x \notin I$: the sequence of stalks is trivial,
- $x \in I_1 \setminus J$: we find the sequence of stalks $0 \to \mathbf{k} \xrightarrow{\alpha \mathrm{id}} \mathbf{k} \to 0$ which is exact,
- $x \in I_2 \setminus J$: same argument,
- $x \in J$: we find the sequence of stalks $0 \to \mathbf{k} \xrightarrow{\binom{\alpha \mathrm{id}}{-\mathrm{id}}} \mathbf{k}^2 \xrightarrow{(\mathrm{id}, \alpha \mathrm{id})} \mathbf{k} \to 0$ which is exact.

Hence the sequence of sheaves is exact, that is, $\mathbf{k}_I = \ker(u_\alpha)$.

(ii-a) We consider two other sections $t_+ \neq 0 \in \Gamma(A_+; F)$ and $t_- \neq 0 \in \Gamma(A_-; F)$ and we define d_0, d_π by the relations between germs:

$$(t_-)_\pi = d_\pi \cdot (t_+)_\pi, \qquad (t_-)_0 = d_0 \cdot (t_+)_0,$$

Since $\Gamma(A_+; F) \simeq \mathbf{k}$ and $s_+, t_+ \neq 0$, there exists $a_+ \neq 0 \in \mathbf{k}$ such that $t_+ = a_+ s_+$. In the same way there exists $a_- \neq 0 \in \mathbf{k}$ such that $t_- = a_- s_-$. We deduce

$$a_-(s_-)_\pi = d_\pi a_+ \cdot (s_+)_\pi, \qquad a_-(s_-)_0 = d_0 a_+ \cdot (s_+)_0.$$

It follows that $c_\pi = (a_+/a_-)d_\pi$ and $c_0 = (a_+/a_-)d_0$ and then that $c_\pi/c_0 = d_\pi/d_0$, as required.

(ii-b) We choose two sections $s_+ \neq 0 \in \Gamma(A_+;F)$ and $s_- \neq 0 \in \Gamma(A_-;F)$ as in the definition of $m(F)$ and take their images by f:

$$u_+ = (f(A_+))(s_+) \in \Gamma(A_+;G), \qquad u_- = (f(A_-))(s_-) \in \Gamma(A_-;G)$$

Taking the germs, we have $(u_+)_\pi = f_\pi((s_+)_\pi)$ and $(u_-)_\pi = f_\pi((s_-)_\pi)$. Applying f_π to the formula $(s_-)_\pi = c_\pi \cdot (s_+)_\pi$, we then find $(u_-)_\pi = c_\pi (u_+)_\pi$. In the same way $(u_-)_0 = c_0(u_+)_0$. Hence $m(G) = c_\pi/c_0 = m(F)$.

(iii) We have

$$F(A_+) = \ker\Big(f_\alpha(A_+)\colon \mathbf{k}_{B_1}(A_+) \oplus \mathbf{k}_{B_2}(A_+) \to \mathbf{k}_{C_+}(A_+) \oplus \mathbf{k}_{C_-}(A_+)\Big).$$

Now $\mathbf{k}_{B_i}(A_+) = \mathbf{k}_{C_+}(A_+) = \mathbf{k}$, $\mathbf{k}_{C_-}(A_+) = 0$ and $f_\alpha(A_+)$ becomes $f_\alpha(A_+) = (\mathrm{id}, \mathrm{id})\colon \mathbf{k}^2 \to \mathbf{k}$. In particular $(1,-1)$ belongs to $\ker(f_\alpha(A_+))$ and defines a section $s_+ \in F(A_+)$.

In the same way $f_\alpha(A_-)$ is identified with $f_\alpha(A_-) = (\mathrm{id}, \alpha\mathrm{id})\colon \mathbf{k}^2 \to \mathbf{k}$. In particular $(\alpha, -1)$ defines a section $s_- \in F(A_-)$.

The sheaf F_α is defined by the exact sequence $0 \to F_\alpha \to \mathbf{k}_{B_1} \oplus \mathbf{k}_{B_2} \to \mathbf{k}_{C_+} \oplus \mathbf{k}_{C_-}$ and taking the germs at π gives the exact sequence $0 \to (F_\alpha)_\pi \to (\mathbf{k}_{B_1})_\pi \to 0$. Hence we have a natural identification $(F_\alpha)_\pi \xrightarrow{\sim} (\mathbf{k}_{B_1})_\pi = \mathbf{k}$. Using this identification we have $(s_+)_\pi = 1$ and $(s_-)_\pi = \alpha$. Hence $c_\pi = \alpha$.

In the same way we have an identification $(F_\alpha)_0 \xrightarrow{\sim} (\mathbf{k}_{B_2})_0 = \mathbf{k}$ which gives $(s_+)_0 = 1$ and $(s_-)_0 = 1$. Hence $c_0 = 1$.

Finally $m(F) = \alpha$.

Locally constant sheaves. Sheaves satisfying (12.1) are called "locally constant". In general a locally constant sheaf with stalk M on a space X is a sheaf F such that any point $x \in X$ has a neighborhood U such that $F|_U \simeq M_U$. Such sheaves are classified by the representations of $\pi_1(X)$ into M (group morphisms $\pi_1(X) \to \mathrm{Aut}(M)$). The representation corresponding to such a sheaf F is called the monodromy of F.

In our case $M = \mathbf{k}$ and $X = S^1$, $\mathrm{Aut}(\mathbf{k}) = \mathbf{k}^\times$ and $\pi_1(S^1) = \mathbb{Z}$. A representation of $\mathbb{Z}$ is the determined by the image of 1. Here it is a scalar $\alpha \in \mathbf{k}^\times$. The F_α defined above is the sheaf with monodromy α.

14. More general resolutions

Let $\mathcal{C}$, $\mathcal{C}'$ be abelian categories. We assume that $\mathcal{C}$ has enough injectives. For a given left exact functor $F\colon \mathcal{C} \to \mathcal{C}'$ we may compute RF with more general resolutions than injective resolutions.

Definition 14.1. An object $X \in \mathcal{C}$ such that $R^iF(X) \simeq 0$ for all $i \geq 1$ is called *F-acyclic.*

Lemma 14.2. *Let $0 \to X^0 \to X^1 \to \cdots \to X^n \to 0$ be an exact sequence in $\mathcal{C}$. We assume that $X^0, \dots, X^{n-1}$ are F-acyclic. Then X^n is F-acyclic.*

Proof. We proceed by induction on n. The case $n = 2$ is true by the long exact sequence $\cdots \to R^iF(X^1) \to R^iF(X^2) \to R^{i+1}F(X^0) \to \cdots$.

For $n > 2$ we split our sequence in two exact sequences

$$0 \to X^0 \to X^1 \to \cdots \to X^{n-2} \to Y^{n-1} \to 0,$$
$$0 \to Y^{n-1} \to X^{n-1} \to X^n \to 0,$$

where $Y^{n-1} = \operatorname{im}(X^{n-2} \to X^{n-1}) \simeq \ker(X^{n-1} \to X^n)$. By induction Y^{n-1} is F-acyclic. Hence X^n is F-acyclic by the case $n = 2$. $\square$

Lemma 14.3. *Let $0 \to X^0 \to X^1 \to \cdots \to X^n \to 0$ be an exact sequence in $\mathcal{C}$ (here n may be ∞). We assume that $X^0, \dots, X^n$ are F-acyclic. Then the sequence $0 \to F(X^0) \to F(X^1) \to \cdots \to F(X^n) \to 0$ is exact.*

Proof. We proceed as in the proof of the previous lemma. The case $n = 2$ is true since $R^1F(X^0) \simeq 0$. In general we split the sequence as in the previous lemma. Then Y^{n-1} is F-acyclic and the induction gives the exact sequences

$$0 \to F(X^0) \to F(X^1) \to \cdots \to F(X^{n-2}) \to F(Y^{n-1}) \to 0,$$
$$0 \to F(Y^{n-1}) \to F(X^{n-1}) \to F(X^n) \to 0,$$

which glue into the exact sequence of the current lemma. $\square$

Proposition 14.4. *Let $X \in \mathrm{Ob}(\mathcal{C})$ and let $J \in \mathbf{C}^+(\mathcal{J})$ be a resolution of X by F-acyclic objects. Then we have an isomorphism $R^iF(X) \simeq H^i\,\mathbf{C}(F)(J)$.*

Proof. (i) As in Proposition 11.3, we can find an injective resolution $I \in \mathbf{C}^+(\mathcal{C})$ of X and a morphism $u\colon J \to I$ in $\mathbf{C}(\mathcal{C})$ which lifts id_X:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & X & \xrightarrow{\varepsilon^0} & J^0 & \xrightarrow{d_J^0} & J^1 & \longrightarrow & J^2 & \longrightarrow & \cdots \\
 & & \| & & \downarrow{\scriptstyle u^0} & & \downarrow{\scriptstyle u^1} & & \downarrow{\scriptstyle u^2} & & \\
0 & \longrightarrow & X & \xrightarrow{\varepsilon^1} & I^0 & \xrightarrow{d_I^0} & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots
\end{array}$$

and such that the u^k are monomorphisms. Indeed we first choose a monomorphism u^0 with I^0 injective and set $\varepsilon^1 = u^0 \circ \varepsilon^0$. This gives the first commutative square. We define $p^0 \colon I^0 \to \operatorname{coker}(\varepsilon^1)$ and $v^0 = p^0 \circ u^0$. We set $J'^1 = \operatorname{coker} \binom{-v^0}{d^0_J}$ so that we have the commutative diagram

$$\begin{array}{ccc} J^0 & \xrightarrow{d^0_J} & J^1 \\ \downarrow{\scriptstyle v^0} & & \downarrow{\scriptstyle w^1} \\ \operatorname{coker}(\varepsilon^1) & \xrightarrow{i^0} & J'^1 \end{array}$$

where i_0 is a monomorphism (check!). We choose a monomorphism $j^1 \colon J'^1 \to I^1$ with I^1 injective and set $d^0_I = j^1 \circ i^0 \circ p^0$, $u^1 = j^1 \circ w^1$. Since $j^1 \circ i^0$ is a monomorphism, we have $\ker(d^0_I) = \ker(p^0) = \operatorname{im}(\varepsilon^1)$. This gives the second commutative square. We go on by induction.

(ii) We set $K = \operatorname{coker}(u)$. Then $0 \to J \to I \to K \to 0$ is a short exact sequence and Proposition 10.7 implies that $H^i K \simeq 0$ for all $i \in \mathbb{Z}$. Hence, viewing K as a long sequence in $\mathcal{C}$, it is an exact long sequence (we say that K is an acyclic complex).

By Lemma 14.2 with $n = 2$, the K^i's are F-acyclic. By Lemma 14.3 we deduce that the long sequence $F(K^\cdot)$ is exact. In other words $H^i(\mathbf{C}(F)(K)) \simeq 0$ for all $i \in \mathbb{Z}$.

By Lemma 14.3 again, with $n = 2$, the sequences $0 \to F(J^i) \to F(I^i) \to F(K^i) \to 0$ are exact. Now the result follows from Proposition 10.7. ☐

Definition 14.5. A family of objects $\mathcal{J} \subset \operatorname{Ob}(\mathcal{C})$ is called F-injective if

(i) for any $X \in \operatorname{Ob}(\mathcal{C})$ there exist $J \in \mathcal{J}$ and a monomorphism $0 \to X \to J$,

(ii) for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$, if $X' \in \mathcal{J}$ and $X \in \mathcal{J}$, then $X'' \in \mathcal{J}$,

(iii) for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$ with $X', X, X'' \in \mathcal{J}$, the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ is exact.

Lemma 14.6. *The objects in $\mathcal{J}$ are F-acyclic.*

Proof. We choose an injective object $I \in \mathcal{C}$ and a monomorphism $a \colon J \to I$. Then we choose $J' \in \mathcal{J}$ and a monomorphism $b \colon I \to J'$.

We set $X = \operatorname{coker}(a)$ and $J'' = \operatorname{coker}(b)$. We have the exact sequences

$$\begin{array}{ccccccccc} 0 & \to & J & \to & I & \to & X & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & J & \to & J' & \to & J'' & \to & 0 \end{array}$$

and the long cohomology exact sequences

$$\begin{array}{ccccccccc} 0 & \to & F(J) & \to & F(I) & \to & F(X) & \to & R^1F(J) & \to & R^1F(I) \\ & & \| & & \downarrow & & \downarrow & & \| & & \downarrow \\ 0 & \to & F(J) & \to & F(J') & \to & F(J'') & \xrightarrow{u} & R^1F(J) & \xrightarrow{v} & R^1F(J'), \end{array}$$

$$\begin{array}{ccccc} R^iF(X) & \to & R^{i+1}F(J) & \to & R^{i+1}F(I) \\ \downarrow & & \| & & \downarrow \\ R^iF(J'') & \to & R^{i+1}F(J) & \xrightarrow{v^{i+1}} & R^{i+1}F(J'), \end{array}$$

Now we prove the result by induction on i.

We first consider $i = 1$. By (ii) of Definition 14.5 we have $J'' \in \mathcal{J}$ and then (iii) implies that the morphism u is zero. Hence v is a monomorphism. Since v factorizes through $R^1F(I)$, which is zero since I is injective, we deduce that $R^1F(J)$ is zero, as claimed.

Assuming the result true for i, we have $R^iF(J'') \simeq 0$ since $J'' \in \mathcal{J}$. Hence the morphism v^{i+1} is a monomorphism and we conclude as in the case $i = 1$ that $R^{i+1}F(J) \simeq 0$. □

We can modify the proof of Prop. 11.3 to obtain:

Proposition 14.7. *If $\mathcal{J}$ is an F-injective family, then any complex $X \in \mathbf{C}^+(\mathcal{C})$ has a resolution by objects of $\mathcal{J}$, that is, there exist a complex of objects in $\mathcal{J}$, $J \in \mathbf{C}^+(\mathcal{J})$, and a morphism $u\colon X \to J$ in $\mathbf{C}^+(\mathcal{C})$ such that u is a qis.*

15. Injectives sheaves

Let X be a topological space and $\mathbf{Sh}(X)$ the category of sheaves of abelian groups. We have seen in Lemma 11.21 that $\mathbf{Ab}$ has enough injectives.

For an abelian group A and $x \in X$ we have the "skyscrapper sheaf" $A_{\{x\}}$ defined by $A_{\{x\}}(U) = A$ if $x \in U$ and $A_{\{x\}}(U) = 0$ else, with the natural restriction maps. For $F \in \mathbf{Sh}(X)$ we have a natural morphism $p_x \colon F \to (F_x)_{\{x\}}$ defined by $p_x(U)(s) = s_x$.

Lemma 15.1. *For any $F \in \mathbf{Sh}(X)$, we have* $\mathrm{Hom}_{\mathbf{Sh}(X)}(F, A_{\{x\}}) \simeq \mathrm{Hom}_{\mathbf{Ab}}(F_x, A)$.

Lemma 15.2. *If A is injective in $\mathbf{Ab}$, then $A_{\{x\}}$ is injective in $\mathbf{Sh}(X)$.*

Lemma 15.3. *The categories $\mathbf{Psh}(X)$ and $\mathbf{Sh}(X)$ admit arbitrary products. Moreover, for a family of presheaves P_i, $i \in I$, the product is given by $(\prod_i P_i)(U) = \prod_i P_i(U)$. If the P_i are sheaves, then $\prod_i P_i$ is also a sheaf and is the product of the P_i's in $\mathbf{Sh}(X)$.*

Lemma 15.4. *Let F_i, $i \in I$, be a family of injective sheaves. Then $\prod_i F_i$ is injective.*

Lemma 15.5. *Let $F \in \mathbf{Sh}(X)$. We define a morphism $i = \prod_{x \in X} p_x \colon F \to \prod_{x \in X} (F_x)_{\{x\}}$ by $i(U)(s) = \prod s_x$. Then i is a monomorphism.*

Proposition 15.6. *The category $\mathbf{Sh}(X)$ has enough injectives.*

Since $\mathbf{Sh}(X)$ has enough injectives we can derive all left exact functors. Important examples are the following functors, for a given $F \in \mathbf{Sh}(X)$ and a given $U \in \mathrm{Op}(X)$,

$$\mathrm{Hom}(F, -) \colon \mathbf{Sh}(X) \to \mathbf{Ab} \qquad \Gamma(U; -) \colon \mathbf{Sh}(X) \to \mathbf{Ab}$$
$$G \mapsto \mathrm{Hom}(F, G) \qquad F \mapsto \Gamma(U; F)$$

There are special notations for their derived functors:

$$\mathrm{Ext}^i(F, G) = \mathrm{R}^i\mathrm{Hom}(F, G), \tag{15.1}$$

$$H^i(U; F) = \mathrm{R}^i\Gamma(U; F). \tag{15.2}$$

The functor $\mathrm{Hom}(F, -)$ is defined for any abelian category and the notation $\mathrm{Ext}^i(-, -)$ is also used for any abelian category.

16. Adjoint functors

Let $\mathcal{C}$, $\mathcal{C}'$ be categories and let $R\colon \mathcal{C}' \to \mathcal{C}$, $L\colon \mathcal{C} \to \mathcal{C}'$ be two functors. Roughly speaking, we say that L is left adjoint to R if $\mathrm{Hom}_{\mathcal{C}}(X, R(Y)) \simeq \mathrm{Hom}_{\mathcal{C}'}(L(X), Y)$ for all $X \in \mathcal{C}$, $Y \in \mathcal{C}'$. Of course we want these isomorphisms to be functorial in X and Y. For this we remark that $\mathrm{Hom}_{\mathcal{C}}(\cdot,\cdot)$ is a functor from $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ to **Set**. In the same way $\mathrm{Hom}_{\mathcal{C}}(\cdot, R(\cdot))$ and $\mathrm{Hom}_{\mathcal{C}'}(L(\cdot),\cdot)$ are functors from $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}'$ to **Set**. Now we can give a formal definition:

Definition 16.1. Let $\mathcal{C}$, $\mathcal{C}'$ be categories and let $R\colon \mathcal{C}' \to \mathcal{C}$, $L\colon \mathcal{C} \to \mathcal{C}'$ be two functors. We say that L is left adjoint to R (or R right adjoint to L, or (L, R) is an adjoint pair) if there exists an isomorphism of functors from $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}'$ to **Set**:

$$\mathrm{Hom}_{\mathcal{C}}(\cdot, R(\cdot)) \simeq \mathrm{Hom}_{\mathcal{C}'}(L(\cdot), \cdot). \tag{16.1}$$

It is called the adjunction morphism.

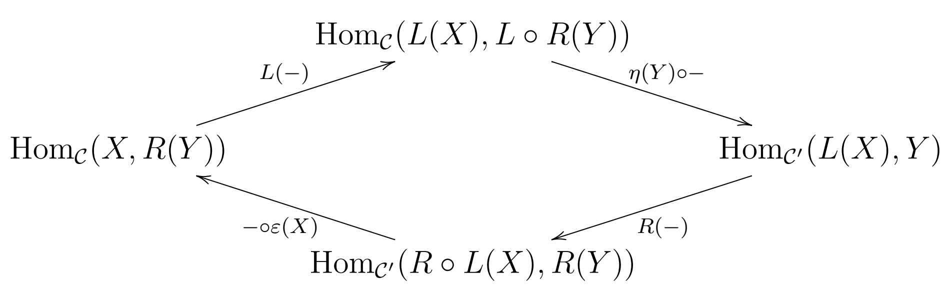
Lemma 16.2. *let $F\colon \mathcal{C} \to \mathcal{C}'$ be a functor. If F has a right (or left) adjoint, then this adjoint is unique, up to a canonical isomorphism.*

Proof. Let G, G' be two right adjoints. For any $X \in \mathcal{C}$, $Y \in \mathcal{C}'$ we have $\mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \simeq \mathrm{Hom}_{\mathcal{C}'}(F(X), Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X, G'(Y))$. Setting $X = G(Y)$ the image of $\mathrm{id}_{G(Y)}$ gives $\theta(Y)\colon G(Y) \to G'(Y)$. Using the functoriality we see that θ is a morphism of functors. Switching G, G' gives $\theta'\colon G' \to G$. By construction the composition $\theta' \circ \theta$ gives the identity morphism $\mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \to \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$, $\forall X, Y$, and it follows that $\theta' \circ \theta = \mathrm{id}$. $\square$

Setting $X = R(Y)$ in the equality $\mathrm{Hom}_{\mathcal{C}}(X, R(Y)) \simeq \mathrm{Hom}_{\mathcal{C}'}(L(X), Y)$ the image of $\mathrm{id}_{R(Y)}$ gives $\eta(Y)\colon L \circ R(Y) \to Y$. As in the proof of Lemma 16.2 we can see η is a morphism of functors. Setting $Y = L(X)$ gives a morphism in the other direction. So we obtain

$$\varepsilon\colon \mathrm{id}_{\mathcal{C}} \to R \circ L, \qquad \eta\colon L \circ R \to \mathrm{id}_{\mathcal{C}'} \tag{16.2}$$

and we can check that the bijection (16.1) is given as the compositions



Runing around this diagram gives the identity morphisms at left and right hand sides. Setting $Y = L(X)$ or $X = R(Y)$ we deduce that the following compositions are the identity morphisms:

$$(\eta \circ L) \circ (L \circ \varepsilon) = \mathrm{id}_L \colon L \to L \circ R \circ L \to L, \tag{16.3}$$

$$(R \circ \eta) \circ (\varepsilon \circ R) = \mathrm{id}_R \colon R \to R \circ L \circ R \to R. \tag{16.4}$$

We can prove (see for example [2] Prop. 1.5.4):

Lemma 16.3. *If L, R are functors and ε, η morphisms of functors satisfying* (16.3), (16.4)*, then (L, R) is an adjoint pair.*

Example 16.4. Let $for\colon \mathbf{Ab} \to \mathbf{Set}$ be the forgetful functor. Then for has a left adjoint, the "free abelian group" functor $I \mapsto \mathbb{Z}^{(I)}$, that is, $\mathrm{Hom}_{\mathbf{Set}}(I, for(A)) \simeq \mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}^{(I)}, A)$.

In the same way we can define the functors "free $\mathbf{k}$-module" for a ring $\mathbf{k}$, "free group", "free associative $\mathbf{k}$-algebra",...

Example 16.5. Let X be a topological space. The forgetful functor $for\colon \mathbf{Sh}(X) \to \mathbf{Psh}(X)$ and the "associated sheaf functor" $\mathbf{Psh}(X) \to \mathbf{Sh}(X)$, $P \mapsto P^a$ are adjoint: we have the functorial isomorphism

$$\mathrm{Hom}_{\mathbf{Psh}(X)}(P, for(F)) \simeq \mathrm{Hom}_{\mathbf{Sh}(X)}(P^a, F)$$

for $P \in \mathbf{Psh}(X)$, $F \in \mathbf{Sh}(X)$.

Exercise 16.6. Let $\mathcal{C}$, $\mathcal{C}'$ be abelian categories and let $R\colon \mathcal{C}' \to \mathcal{C}$, $L\colon \mathcal{C} \to \mathcal{C}'$ be additive functors such that R is right adjoint to L. Prove that R is left exact and L is right exact.

17. Direct and inverse images of sheaves

Let $f\colon X \to Y$ be a continuous map between topological spaces. Let $\mathbf{k}$ be a ring. We define the direct image functor $f_*\colon \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ and the inverse image functor $f^{-1}\colon \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$. They form an adjoint pair (f^{-1}, f_*). When X and Y are Hausdorff and locally compact we also define the proper direct image functor $f_!\colon \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$.

Definition 17.1. For $F \in \mathbf{Psh}(X)$ we define $f_*F \in \mathbf{Psh}(Y)$ by its sections $(f_*F)(V) = F(f^{-1}(V))$ for any open subset $V \subset Y$, with the restriction maps naturally given by those of F. If $F \in \mathbf{Sh}(X)$ we can check that $f_*F \in \mathbf{Sh}(Y)$.

If $u\colon F \to G$ is a morphism in $\mathbf{Sh}(X)$, we define $f_*u\colon f_*F \to f_*G$ by $(f_*u)(V) = u(f^{-1}(V))$. We obtain functors $f_*\colon \mathbf{Psh}(X) \to \mathbf{Psh}(Y)$, $f_*\colon \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$.

Lemma 17.2. *The functor $f_*\colon \mathbf{Psh}(X) \to \mathbf{Psh}(Y)$ is exact and the functor $f_*\colon \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ is left exact.*

Definition 17.3. For $G \in \mathbf{Psh}(Y)$ we define a presheaf ${}^{\mathrm{pr}}f^{-1}G \in \mathbf{Psh}(X)$ by $({}^{\mathrm{pr}}f^{-1}G)(U) = \varinjlim_{V \supset f(U)} G(V)$, where V runs over the open neighborhoods of $f(U)$ in Y. The restriction maps are naturally induced by those of G. A morphism $u\colon F \to G$ induces morphisms on the inductive limits, $({}^{\mathrm{pr}}f^{-1}u)(U)\colon ({}^{\mathrm{pr}}f^{-1}F)(U) \to ({}^{\mathrm{pr}}f^{-1}G)(U)$, for all $U \in \mathrm{Op}(X)$, which are compatible and define ${}^{\mathrm{pr}}f^{-1}u\colon {}^{\mathrm{pr}}f^{-1}F \to {}^{\mathrm{pr}}f^{-1}G$. This gives a functor ${}^{\mathrm{pr}}f^{-1}\colon \mathbf{Psh}(X) \to \mathbf{Psh}(Y)$.

We set $f^{-1}G = ({}^{\mathrm{pr}}f^{-1}G)^a$ and obtain a functor $f^{-1}\colon \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$.

When $f\colon X \to Y$ is an embedding of topological spaces (that is, f is an inclusion and the topology of X is the induced topology) we often write

$$G|_X := f^{-1}G. \tag{17.1}$$

If f is the inclusion of an open set, we have $(G|_X)(U) = G(U)$, for all $U \in \mathrm{Op}(X)$.

Exercise 17.4. Let X be a Hausdorff topological space and $Z \subset X$ a compact subset. Then, for any $F \in \mathbf{Sh}(X)$ and $V \in \mathrm{Op}(Z)$, we have $(F|_Z)(V) \simeq \varinjlim_{U \supset V} F(U)$, where U runs over the open neighborhoods of V in X.

Lemma 17.5. *Let $f\colon X \to Y$ be a continuous map and let $x \in Y$. For any $P \in \mathbf{Psh}(Y)$ or $F \in \mathbf{Sh}(Y)$, we have natural isomorphisms $({}^{\mathrm{pr}}f^{-1}P)_x \simeq P_{f(x)}$, $(f^{-1}F)_x \simeq F_{f(x)}$.*

Since the exactness of a sequence of sheaves can be checked in the stalks we deduce:

Lemma 17.6. *For any continuous map $f\colon X \to Y$, the functor f^{-1} is exact.*

In the situation of Definitions 17.1 and 17.3 we define two morphisms of functors

$$\varepsilon\colon \mathrm{id}_{\mathbf{Psh}(Y)} \to f_* \circ {}^{\mathrm{pr}}f^{-1}, \qquad \eta\colon {}^{\mathrm{pr}}f^{-1} \circ f_* \to \mathrm{id}_{\mathbf{Psh}(X)} \tag{17.2}$$

as follows. For $G \in \mathbf{Psh}(Y)$ and $V \in \mathrm{Op}(Y)$ we have

$$f_* \circ {}^{\mathrm{pr}}f^{-1}(G)(V) = ({}^{\mathrm{pr}}f^{-1}G)(f^{-1}V) = \varinjlim_{W} G(W),$$

where $W \subset Y$ runs over the open subsets such that $f(f^{-1}(V)) \subset W$. We remark that V belongs to this family of W's. Hence we have a natural morphism $G(V) \to f_* \circ {}^{\mathrm{pr}}f^{-1}(G)(V)$. It is compatible with the restrictions maps for $V' \subset V$ and gives $\varepsilon(G)\colon G \to f_* \circ {}^{\mathrm{pr}}f^{-1}(G)$. For any $F \in \mathbf{Psh}(X)$ and $U \in \mathrm{Op}(X)$ we have

$${}^{\mathrm{pr}}f^{-1} \circ f_*(F)(U) = \varinjlim_{W} F(f^{-1}(W)),$$

where $W \subset X$ runs over the open subsets such that $f(U) \subset W$, that is, $U \subset f^{-1}(W)$. We thus have a natural morphism ${}^{\mathrm{pr}}f^{-1} \circ f_*(F)(U) \to F(U)$, which defines our η.

We can check the hypothesis of Lemma 16.3 and deduce that $({}^{\mathrm{pr}}f^{-1}, f_*)$ is an adjoint pair. Using Example 16.5 we obtain:

Proposition 17.7. *Let $f\colon X \to Y$ be a continuous map between topological spaces. The pairs of functors $({}^{\mathrm{pr}}f^{-1}, f_*)$ and (f^{-1}, f_*) are adjoint pairs.*

Definition 17.8. Let $F \in \mathbf{Sh}(X)$, $U \in \mathrm{Op}(X)$ and $s \in F(U)$. The support of s is the closed subset $\mathrm{supp}(s)$ of U defined by

$$U \setminus \mathrm{supp}(s) = \bigcup_{V \in \mathrm{Op}(U),\, s|_V \simeq 0} V.$$

Alternatively $U \setminus \mathrm{supp}(s)$ is the biggest open subset V of U such that $s|_V \simeq 0$.

A topological space X is locally compact if, for any $x \in X$ and any neighborhood U of x, there exists a compact neighborhood of x contained in U. Now we assume X, Y are Hausdorff and locally compact. Then a map $f\colon X \to Y$ is *proper* if the inverse image of any compact subset of Y is compact.

Definition 17.9. Let $f\colon X \to Y$ be a continuous map of Hausdorff and locally compact spaces. For $F \in \mathbf{Sh}(X)$ we define a subsheaf $f_!F \in \mathbf{Sh}(Y)$ of f_*F by

$$(f_!F)(V) = \{s \in (f^{-1}(V));\ f|_{\operatorname{supp} s}\colon \operatorname{supp}(s) \to V \quad \text{is proper}\}$$

for any open subset $V \subset Y$. If $u\colon F \to G$ is a morphism in $\mathbf{Sh}(X)$, the morphism $f_*u\colon f_*F \to f_*G$ sends $f_!F$ to $f_!G$. We obtain a functor $f_!\colon \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$.

Remark 17.10. If the map f itself is proper, then we have $f_! \xrightarrow{\sim} f_*$.

Definition 17.11. Let X be a Hausdorff and locally compact topological space. For $F \in \mathbf{Sh}(X)$ and $U \in \operatorname{Op}(X)$ we set

$$\Gamma_c(U; F) = \{s \in F(V);\ \operatorname{supp}(s) \text{ is compact.}\}$$

Proposition 17.12. *Let $f\colon X \to Y$ be as in Definition 17.9. For any $F \in \mathbf{Sh}(X)$ and $y \in Y$ we have*

$$(f_!F)_y \simeq \Gamma_c(f^{-1}(y); F|_{f^{-1}(y)}).$$

Remark 17.13. The functors of global sections $\Gamma(X; -)$ and global sections with compact support $\Gamma_c(X; -)$ are particular cases of f_*, $f_!$. Indeed we can identify $\mathbf{Sh}(\mathrm{pt})$ with $\mathbf{Ab}$, and, for $a\colon X \to \mathrm{pt}$, the map to a point, we have

$$a_*(F) \simeq \Gamma(X; F), \qquad a_!(F) \simeq \Gamma_c(X; F).$$

We can recover the sheaf $\mathbf{k}_W$ of Exercise 8.4.

Notation 17.14. Let X be a locally compact Hausdorff space. Let $j\colon W \to X$ be the inclusion of a locally closed subset (we can write $W = V \cap Z$, with V open and Z closed). Let A be an abelian group and let $A_W \in \mathbf{Sh}(W)$ be the constant sheaf on W defined in Example 3.11. We define

$$A_{X,W} = j_!A_W.$$

When it is clear that the ambient space is X, we simply write A_W instead of $A_{X,W}$.

Exercise 17.15. Check that $A_{X,W}$ is indeed the sheaf of Exercise 8.4 (for $A = \mathbf{k}$).

Proposition 17.12 gives

$$(A_W)_x \simeq \begin{cases} A & \text{if } x \in W, \\ 0 & \text{if } x \notin W. \end{cases}$$

Lemma 17.16. *Let $U \subset X$ be an open subset and $F \in \mathbf{Sh}(X)$. Let $(\mathbb{1}_U \in \mathbb{Z}_U(U)$ be the constant function 1. The morphism*

$$\mathrm{Hom}(\mathbb{Z}_U, F) \to \Gamma(U; F), \qquad \varphi \mapsto ((\varphi(U))(U))(\mathbb{1}_U)$$

is an isomorphism.

Proof. Exercise (try to define an inverse.) □

The morphism of the lemma is functorial in F and we have in fact an isomorphism of functors

$$\mathrm{Hom}(\mathbb{Z}_U, -) \to \Gamma(U; -).$$

A useful consequence of this lemma is the isomorphism of derived functors (recall the notations (15.1) and (15.2)):

$$H^i(U; F) \simeq \mathrm{Ext}^i(\mathbb{Z}_U, F). \tag{17.3}$$

Lemma 17.17. *Let $U_1, U_2 \in \mathrm{Op}(X)$ and set $U_{12} = U_1 \cap U_2$ and $U = U_1 \cup U_2$. We have an exact sequence*

$$0 \to \mathbb{Z}_{U_{12}} \xrightarrow{u} \mathbb{Z}_{U_1} \oplus \mathbb{Z}_{U_2} \xrightarrow{v} \mathbb{Z}_U \to 0,$$

where $u = \binom{u_1}{u_2}$, $v = (v_1, -v_2)$ and $u_k \colon \mathbb{Z}_{U_{12}} \to \mathbb{Z}_{U_k}$, $v_k \colon \mathbb{Z}_{U_k} \to \mathbb{Z}_U$ are the natural morphisms.

Proof. Same proof as Exercise 12.2 (see §13). □

Lemma 17.18 (Mayer-Vietoris)**.** *We use the notations of Lemma 17.17. Let $F \in \mathbf{Sh}(X)$. Then we have a long exact sequence*

$$\cdots \to H^i(U; F) \to H^i(U_1; F) \oplus H^i(U_2; F) \to H^i(U_{12}; F) \\ \to H^{i+1}(U; F) \to \cdots$$

Proof. By (17.3) we can replace $H^i(U; F)$ by $\mathrm{Ext}^i(\mathbb{Z}_U, F)$. To compute these functors we replace F by an injective resolution, say (I^*, d^*). We apply the functors $\mathrm{Hom}(-, I^k)$ to the sequence of Lemma 17.17 and we find a sequence of complexes:

$$\begin{array}{ccccccccc} & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}(\mathbb{Z}_U, I^k) & \longrightarrow & \bigoplus_{i=1}^{2} \mathrm{Hom}(\mathbb{Z}_{U_i}, I^k) & \longrightarrow & \mathrm{Hom}(\mathbb{Z}_{U_{12}}, I^k) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathrm{Hom}(\mathbb{Z}_U, I^{k+1}) & \to & \bigoplus_{i=1}^{2} \mathrm{Hom}(\mathbb{Z}_{U_i}, I^{k+1}) & \to & \mathrm{Hom}(\mathbb{Z}_{U_{12}}, I^{k+1}) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \end{array}$$

The rows are exact because each I^k is injective. Hence we have a short exact sequence of complexes. By (17.3) the cohomology of the columns give $H^*(U;F)$, $\oplus_i H^*(U_i;F)$,... Now the lemma follows from Proposition 10.7. □

Lemma 17.19. *Let $f\colon X \to Y$, $g\colon Y \to Z$ be continuous map. We have $(g\circ f)_* \simeq g_* \circ f_*$ and $(g\circ f)^{-1} \simeq f^{-1}\circ g^{-1}$. If X, Y, Z are Hausdorff locally compact we also have $(g\circ f)_! \simeq g_! \circ f_!$.*

Proof. The first isomorphism follows quickly from the definiition of $(-)_*$. The second one follows from the first by uniqueness of adjoint functor. The third one follows from the first by checking that the supports are proper. □

Lemma 17.20. *Let $f\colon X \to Y$ be a continuous map and $a_Y\colon Y \to \mathrm{pt}$ be the map to a point. Then for any group $M \in \mathbf{Ab}$ we have $M_Y \simeq a_Y^{-1}(M)$ (using $\mathbf{Sh}(\mathrm{pt}) \simeq \mathbf{Ab}$) and $M_X \simeq f^{-1}M_Y$.*

Proof. We see that ${}^{\mathrm{pr}}a_Y^{-1}(M)$ is the constant presheaf PM_Y and we deduce $M_Y \simeq a_Y^{-1}(M)$. Then $f^{-1}M_Y \simeq f^{-1}a_Y^{-1}(M) \simeq a_X^{-1}(M) \simeq M_X$ by Lemma 17.19. □

18. Flabby and soft sheaves

Let X be a topological space.

Definition 18.1. A sheaf $F \in \mathbf{Sh}(X)$ is flabby if, for any open subset $U \subset X$, the restriction morphism $F(X) \to F(U)$ is surjective.

Let $f\colon X \to Y$ be a continuous map.

Proposition 18.2 (see [1], Section 2.4)**.** *The family of flabby sheaves is f_*-injective and $f_!$-injective.*

We apply this proposition to the computation of $H^i(\mathbb{R};\mathbf{k}_{[a,b]})$ for a closed interval $[a,b]$ of $\mathbb{R}$.

We recall the morphism (16.2) $\varepsilon\colon \mathbf{k}_{[a,b]} \to i_*i^{-1}\mathbf{k}_{[a,b]}$, where $i\colon \mathbb{R}_{disc} \to \mathbb{R}$ is the map from $\mathbb{R}$ with the discrete topology to $\mathbb{R}$. We can identify $i_*i^{-1}\mathbf{k}_{[a,b]}$ with the sheaf $\mathcal{F}_{[a,b]}$ of functions on $[a,b]$ defined by $\mathcal{F}_{[a,b]}(U) = \{f\colon U \cap [a,b] \to \mathbb{R}\}$. This sheaf is flabby since we can extend a function defined on $U \cap [a,b]$ arbitrarily to a function defined on $[a,b]$. We define $G = \operatorname{coker}(\varepsilon)$ and we have the short exact sequence:

$$0 \to \mathbf{k}_{[a,b]} \to \mathcal{F}_{[a,b]} \to G \to 0. \tag{18.1}$$

Lemma 18.3. *For any open subset $U \subset \mathbb{R}$ the sequence* (18.1) *gives the exact sequence of sections:*

$$0 \to \Gamma(U;\mathbf{k}_{[a,b]}) \xrightarrow{a(U)} \Gamma(U;\mathcal{F}_{[a,b]}) \xrightarrow{b(U)} \Gamma(U;G) \to 0. \tag{18.2}$$

Proof. (i) Writing U as a disjoint union of intervals, $U = \bigsqcup_{k\in\mathbb{Z}} I_k$, we have $\Gamma(U;F) \simeq \prod_{k\in\mathbb{Z}} \Gamma(I_k;F)$. Since a product of exact sequences of abelian groups is exact, we can assume that U is an interval. We have to check that the last morphism in (18.2) is surjective. Let $s \in \Gamma(U;G)$ be given.

(ii) Let us first prove that for any compact subinterval $K = [c,d] \subset U$ there exists a neighborhood V of K and $s' \in \Gamma(V;\mathcal{F}_{[a,b]})$ such that $b(V)(s') = s|_V$.

For any $x \in U$ there exist a neighborhood W_x of x and $s'(x) \in \Gamma(W_x;\mathcal{F}_{[a,b]})$ such that $b(W_x)(s'(x)) = s|_{W_x}$. We can assume that the W_x are intervals and we choose a finite number of them to cover K. We denote them $W_1,\dots,W_N$ and order them so that $V_i := W_1 \cup \cdots \cup W_i$ is connected, for all i. We also write $s'_i \in \Gamma(W_i;\mathcal{F}_{[a,b]})$ instead of $s'(x)$. Let us prove by induction on i that there exists $s''_i \in \Gamma(V_i;\mathcal{F}_{[a,b]})$ such that $b(V_i)(s''_i) = s|_{V_i}$. For $i = 1$ we have $s''_1 = s'_1$. Assuming s''_i is defined we have

$$b(V_i \cap W_{i+1})(s''_i|_{V_i\cap W_{i+1}}) = s|_{V_i\cap W_{i+1}} = b(V_i \cap W_{i+1})(s'_{i+1}|_{V_i\cap W_{i+1}}).$$

Hence there exists $t_i \in \Gamma(V_i \cap W_{i+1}; \mathbf{k}_{[a,b]})$ such that

$$s''_i|_{V_i \cap W_{i+1}} - s'_{i+1}|_{V_i \cap W_{i+1}} = a(V_i \cap W_{i+1})(t_i).$$

We can extend t_i to a section $t'_i \in \Gamma(W_{i+1}; \mathbf{k}_{[a,b]})$ because $V_i \cap W_{i+1}$ is connected. We set $\tilde{s}_{i+1} = s'_{i+1} - a(W_{i+1})(t'_i)$. Then we see that $s''_i|_{V_i \cap W_{i+1}} = \tilde{s}_i|_{V_i \cap W_{i+1}}$ and we can glue these sections in a section s''_{i+1} such that $b(V_{i+1})(s''_{i+1}) = s|_{V_{i+1}}$.

(iii) We remark that the section $s' \in \Gamma(V; \mathcal{F}_{[a,b]})$ found in (ii) is unique up to the addition of a section of $\Gamma(V; \mathbf{k}_{[a,b]})$, that is, up to the addition of a constant function. Hence, for a given $x_0 \in U$ with $x_0 \in K$, there exists a unique $s' \in \Gamma(V; \mathcal{F}_{[a,b]})$ such that $b(V)(s') = s|_V$ and $s'(x_0) = 0$.

Now we consider an increasing sequence $K_i \subset K_{i+1} \subset U$, $i \in \mathbb{N}$, of compact intervals whose union is U. For each i we have a unique $s'_i \in \Gamma(V_i; \mathcal{F}_{[a,b]})$ such that $b(V_i)(s'_i) = s|_{V_i}$ and $s'_i(x_0) = 0$, where V_i is some neighborhood of K_i. By unicity we have $s'_{i+1}|_{V_i} = s'_i$. Hence we can glue the s'_i in a section $s' \in \Gamma(U; \mathcal{F}_{[a,b]})$ such that $b(U)(s') = s$. □

Lemma 18.4. *The sheaf G of* (18.1) *is flabby.*

Proof. Let $U \subset \mathbb{R}$ and $s \in G(U)$ be given. By Lemma 18.3 there exists $s' \in \mathcal{F}_{[a,b]}(U)$ such that $b(U)(s') = s$. Since $\mathcal{F}_{[a,b]}$ is flabby, there exists $t' \in \mathcal{F}_{[a,b]}(\mathbb{R})$ such that $t'|_U = s'$. Then $t = b(\mathbb{R})(t')$ satisfies $t|_U = s$. □

Hence (18.1) gives a flabby resolution of $\mathbf{k}_{[a,b]}$. We deduce that for any open subset U of $\mathbb{R}$

$$H^i(U; \mathbf{k}_{[a,b]}) \simeq H^i(0 \to \Gamma(U; \mathcal{F}_{[a,b]}) \xrightarrow{b(U)} \Gamma(U; G) \to 0).$$

By Lemma 18.3 the morphism $b(U)$ is surjective and we obtain that the cohomology of $\mathbf{k}_{[a,b]}$ is concentrated in degree 0:

Proposition 18.5. *Let $[a,b]$ be a closed interval in $\mathbb{R}$. For any open interval U of $\mathbb{R}$ such that $U \cap [a,b] \neq \emptyset$, we have*

$$H^0(U; \mathbf{k}_{[a,b]}) \simeq \mathbf{k} \qquad \text{and} \qquad H^i(U; \mathbf{k}_{[a,b]}) \simeq 0 \quad \text{for } i \neq 0.$$

Now we assume X is Hausdorff and locally compact.

Definition 18.6. A sheaf $F \in \mathbf{Sh}(X)$ is c-soft if, for any compact subset $C \subset X$, the restriction morphism $F(X) \to F(C)$ is surjective, where $F(C) := \varinjlim_{C \subset U} F(U)$, U running over the open neighborhoods of C.

We remark that a flabby sheaf is c-soft.

An important example is the case where X is a C^∞ manifold and $F = C^\infty_X$ is the sheaf of C^∞ functions on X. More generally any sheaf of modules over C^∞_X is c-soft, in particular the sheaf of i-forms Ω^i_X is c-soft.

Let $f\colon X \to Y$ be a continuous map, X, Y Hausdorff and locally compact.

Proposition 18.7 (see [1], Section 2.5)**.** *The family of c-soft sheaves is f_*-injective and $f_!$-injective.*

Corollary 18.8. *Let X be a C^∞ manifold. Then $H^i(X;\mathbb{R}_X) \simeq H^i_{dR}(X)$, where $H^i_{dR}(X)$ is the de Rham cohomology of X.*

Proof. By the Poincaré lemma the de Rham complex $0 \to \Omega^0_X \to \Omega^1_X \to \cdots \to \Omega^n_X \to 0$, $n = \dim X$, is a c-soft resolution of $\mathbb{R}_X$. The result follows. □

19. Internal Hom and tensor product

Let X be a topological space.

Definition 19.1. For $F, G \in \mathbf{Sh}(X)$ we define an "internal Hom sheaf" $\mathcal{H}om(F,G) \in \mathbf{Sh}(X)$ as follows. For $U \in \mathrm{Op}(X)$ we set

$$\mathcal{H}om(F,G)(U) = \mathrm{Hom}_{\mathbf{Sh}(U)}(F|_U, G|_U)$$

where $F|_U$ is defined in (17.1) (the restriction to an open set is very simple: $(F|_U)(V) = F(V)$, for $V \in \mathrm{Op}(U)$). For $U' \subset U$ we have $(F|_U)|_{U'} = F|_{U'}$ and we deduce restriction maps $\mathcal{H}om(F,G)(U) \to \mathcal{H}om(F,G)(U')$. We can check that they turn $\mathcal{H}om(F,G)$ into a presheaf and then (exercise!) that $\mathcal{H}om(F,G)$ is in fact a sheaf.

By definition we have $\Gamma(X; \mathcal{H}om(F,G)) = \mathrm{Hom}(F,G)$.

Definition 19.2. For $F, G \in \mathbf{Sh}(X)$ we define a presheaf $F \overset{\mathrm{pr}}{\otimes} G$ by $(F \overset{\mathrm{pr}}{\otimes} G)(U) = F(U) \otimes_{\mathbb{Z}} G(U)$. The restriction maps of F, G give restriction maps for $F \overset{\mathrm{pr}}{\otimes} G$. We set $F \otimes G = (F \overset{\mathrm{pr}}{\otimes} G)^a$.

More generally, if we work with sheaves of R-modules for some *commutative* ring R, we define $F \otimes_R G$ in the same way, starting with $F(U) \otimes_R G(U)$ instead of $F(U) \otimes_{\mathbb{Z}} G(U)$.

We recal the adjunction for R-modules, for a commutative ring R and $M, N, P \in \mathrm{Mod}(R)$,

$$\mathrm{Hom}_R(M \otimes_R N, P) \simeq \mathrm{Hom}_R(M, \mathrm{Hom}_R(N,P)).$$

We can deduce the similar fact for sheaves:

Lemma 19.3. *For a given $G \in \mathbf{Sh}(X,R)$ (the category of sheaves of R-modules), $-\otimes G$ is left adjoint to $\mathcal{H}om(G,-)$. Explicitly, for any $F, G, H \in \mathbf{Sh}(X,R)$:*

$$\mathrm{Hom}_{\mathbf{Sh}(X,R)}(F \otimes_R G, H) \simeq \mathrm{Hom}_{\mathbf{Sh}(X,R)}(F, \mathcal{H}om(G,H)).$$

20. Definition of derived categories

In this section we give a first introduction to derived categories. We only give a brief account on the subject and refer to the first chapter of [1] (or Chapters 10-13 of [2]) for details and proofs.

Definition 20.1. Let $\mathcal{C}$ be an abelian category and let $u\colon X \to Y$ be a morphism in $\mathbf{C}(\mathcal{C})$ or in $\mathbf{K}(\mathcal{C})$. We say that u is a quasi-isomorphism (qis for short) if the morphisms $H^i(u)\colon H^i(X) \to H^i(Y)$ are isomorphisms, for all $i \in \mathbb{Z}$.

A related notion is that of acyclic complexes: a complex X in $\mathbf{C}(\mathcal{C})$ or in $\mathbf{K}(\mathcal{C})$ is *acyclic* (or *exact*) if $H^i(X) \simeq 0$ for all $i \in \mathbb{Z}$ (in other words the long sequence $\cdots X^i \xrightarrow{d^i} X^{i+1} \cdots$ is exact).

Exercise 20.2. Let $u\colon X \to Y$ be a morphism in $\mathbf{C}(\mathcal{C})$. Then u is a qis if and only if $\ker(u)$ and $\operatorname{coker}(u)$ are acyclic.

The derived category of $\mathcal{C}$, denoted $\mathbf{D}(\mathcal{C})$, is obtained from $\mathbf{C}(\mathcal{C})$ by inverting the qis. This process is called *localization*.

Definition 20.3. Let $\mathcal{A}$ be a category and $\mathcal{S}$ a family of morphisms in $\mathcal{A}$. A localization of $\mathcal{A}$ by $\mathcal{S}$ is a category $\mathcal{A}_\mathcal{S}$ (a priori in a bigger universe) and a functor $Q\colon \mathcal{A} \to \mathcal{A}_\mathcal{S}$ such that

(i) for all $s \in \mathcal{S}$, $Q(s)$ is an isomorphism,
(ii) for any category $\mathcal{B}$ and any functor $F\colon \mathcal{A} \to \mathcal{B}$ such that $F(s)$ is an isomorphism for all $s \in \mathcal{S}$, there exists a functor $F_\mathcal{S}\colon \mathcal{A}_\mathcal{S} \to \mathcal{B}$ such that $F \simeq F_\mathcal{S} \circ Q$,
(iii) denoting by $\mathbf{Func}(\cdot,\cdot)$ the category of functors, the functor $\circ Q\colon \mathbf{Func}(\mathcal{A}_\mathcal{S}, \mathcal{B}) \to \mathbf{Func}(\mathcal{A}, \mathcal{B})$ is fully faithful (which implies unicity of $F_\mathcal{S}$ in (ii)).

It is possible to construct $\mathcal{A}_\mathcal{S}$ as a category with the same objects as $\mathcal{A}$ and with morphisms defined as chains $(s_1, u_1, s_2, u_2, \dots, s_n, u_n)$ with $s_i \in \mathcal{S}$ and u_i any morphism in $\mathcal{A}$ and compatible sources/targets ($X_1 \xleftarrow{s_1} Y_1 \xrightarrow{u_1} X_2 \xleftarrow{s_2} Y_2 \xrightarrow{u_2} X_3 \cdots$) modulo some equivalence relation. Such a chain is meant to represent $u_n \circ s_n^{-1} \circ u_{n-1} \circ \cdots \circ s_1^{-1}$. The equivalence relation is generated by $(s_1, u_1, \dots, s_n, u_n) \sim (s_1, u_1, \dots, s, s, \dots, s_n, u_n)$ where (s, s), $s \in \mathcal{S}$, is inserted between u_i and s_{i+1}. The composition is the concatenation.

However in our situation the localization will be obtained by a calculus of fractions and we only need chains (s, u) length 2. We will not use this fact and refer to Section 21.3.

Definition 20.4. Let $\mathcal{C}$ be an abelian category. The derived category of $\mathcal{C}$ is the localization $\mathbf{D}(\mathcal{C}) = (\mathbf{K}(\mathcal{C}))_{Qis}$. We denote by $Q_\mathcal{C}\colon \mathbf{K}(\mathcal{C}) \to$

$\mathbf{D}(\mathcal{C})$ the localization functor (or its composition with $\mathbf{C}(\mathcal{C}) \to \mathbf{K}(\mathcal{C})$). Starting with $\mathbf{K}^*(\mathcal{C})$ where $* = +, -$ or b, we define in the same way $\mathbf{D}^*(\mathcal{C})$.

The obvious functor $\mathbf{C}(\mathcal{C}) \to \mathbf{K}(\mathcal{C})$ sends qis to qis and hence induces a functor $(\mathbf{C}(\mathcal{C}))_{Qis} \to (\mathbf{K}(\mathcal{C}))_{Qis}$. We can prove that this functor is an equivalence (see [**?**]). So we could as well define $\mathbf{D}(\mathcal{C})$ directly from $\mathbf{C}(\mathcal{C})$. The point is that, starting from $\mathbf{K}(\mathcal{C})$, the localization can be constructed by a calculus of fractions.

The categories $\mathbf{K}(\mathcal{C})$ and $\mathbf{D}(\mathcal{C})$ are additive. They are not abelian in general.

By definition the cohomology functors $H^i\colon \mathbf{K}(\mathcal{C}) \to \mathcal{C}$, $i \in \mathbb{Z}$, factorize through the localization functor. We still denote by $H^i\colon \mathbf{D}(\mathcal{C}) \to \mathcal{C}$ the induced functors.

Lemma 20.5. *Let $\mathcal{C}, \mathcal{C}'$ be abelian categories. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be an exact functor. Then $C(F)$ sends acyclic complexes to acyclic complexes and it sends qis to qis. In particular $Q_{\mathcal{C}'} \circ \mathbf{K}(F)\colon \mathbf{K}(\mathcal{C}) \to \mathbf{D}(\mathcal{C}')$ sends qis to isomorphisms and factorizes in a unique way through a functor $\mathbf{D}(\mathcal{C}) \to \mathbf{D}(\mathcal{C}')$ that we still denote by F:*

$$\begin{array}{ccc}
\mathbf{K}(\mathcal{C}) & \xrightarrow{\mathbf{K}(F)} & \mathbf{K}(\mathcal{C}') \\
{\scriptstyle Q_{\mathcal{C}}}\big\downarrow & & \big\downarrow{\scriptstyle Q_{\mathcal{C}'}} \\
\mathbf{D}(\mathcal{C}) & \xrightarrow{F} & \mathbf{D}(\mathcal{C}').
\end{array}$$

We have a natural embedding of $\mathcal{C}$ in $\mathbf{C}(\mathcal{C})$ which sends $X \in \mathcal{C}$ to the complex $(X^{\cdot}, d_X) = \cdots \to 0 \to X \to 0 \to \cdots$ with $X^0 = X$ and $X^i = 0$ for $i \neq 0$. This induces by composition other functors $\mathcal{C} \to \mathbf{K}(\mathcal{C})$ and $\mathcal{C} \to \mathbf{D}(\mathcal{C})$. We can check that all these functors are fully faithful embeddings of $\mathcal{C}$ in $\mathbf{C}(\mathcal{C})$, $\mathbf{K}(\mathcal{C})$ or $\mathbf{D}(\mathcal{C})$.

We have the following generalization of Proposition 11.8.

Proposition 20.6. *Let $\mathcal{C}$ be an abelian category. We assume that $\mathcal{C}$ has enough projectives and we let $\mathcal{P}$ be its full subcategory of projective objects. We denote by $Q|_{\mathcal{P}}\colon \mathbf{K}^-(\mathcal{P}) \to \mathbf{D}^-(\mathcal{C})$ the functor induced by the quotient functor. Then $Q|_{\mathcal{P}}$ is an equivalence of categories.*

Similarly, if $\mathcal{C}$ has enough injectives and $\mathcal{I}$ is the full subcategory of injective objects, then $Q|_{\mathcal{I}}\colon \mathbf{K}^+(\mathcal{I}) \to \mathbf{D}^+(\mathcal{C})$ is an equivalence.

Definition 20.7. Let $\mathcal{C}, \mathcal{C}'$ be abelian categories. We assume that $\mathcal{C}$ has enough projectives. Let $F\colon \mathcal{C} \to \mathcal{C}'$ (or $F\colon \mathbf{C}^-(\mathcal{C}) \to \mathbf{C}^-(\mathcal{C}')$) be a right exact functor. Let $\mathbf{K}(F)\colon \mathbf{K}^-(\mathcal{P}) \to \mathbf{K}^-(\mathcal{C}')$ be the functor induced by F. We define $LF\colon \mathbf{D}^-(\mathcal{C}) \to \mathbf{D}^-(\mathcal{C}')$ by $LF = Q_{\mathcal{C}'} \circ \mathbf{K}(F) \circ \mathbf{res}_{proj}$, where $\mathbf{res}_{proj}$ is an inverse to the equivalence $Q|_{\mathcal{P}}$ of Proposition 20.6.

In the same way, if $\mathcal{C}$ has enough injectives and F is left exact, we define $RF\colon \mathbf{D}^+(\mathcal{C}) \to \mathbf{D}^+(\mathcal{C}')$ by $RF = Q_{\mathcal{C}'} \circ \mathbf{K}(F) \circ \mathbf{res}_{inj}$, with $\mathbf{res}_{inj}$ inverse of $Q|_{\mathcal{I}}$.

By definition we have $H^i LF = L^i F$.

If F is exact then $LF \simeq F \simeq RF$ with the notation of Lemma 20.5.

The first interest of introducing the derived category is the possibility to compose derived functors:

Proposition 20.8. *Let $F\colon \mathcal{C} \to \mathcal{C}'$, $G\colon \mathcal{C}' \to \mathcal{C}''$ be left exact functors between abelian categories. We assume that $\mathcal{C}$ and $\mathcal{C}'$ have enough injectives and that F sends the injective objects of $\mathcal{C}$ to G-acyclic objects of $\mathcal{C}'$. Then $R(G \circ F) \simeq RG \circ RF$.*

Lemma 20.9. *Let $f\colon X \to Y$, $g\colon Y \to Z$, be continuous maps. Then $\mathrm{R}(g \circ f)_* \simeq \mathrm{R}f_* \circ \mathrm{R}g_*$. If the spaces are Hausdorff and locally compact spaces, we also have $\mathrm{R}(g \circ f)_! \simeq \mathrm{R}f_! \circ \mathrm{R}g_!$.*

Proof. We have $(g \circ f)_* \simeq g_* \circ f_*$ by Lemma 17.19. If $F \in \mathbf{Sh}(X)$ is injective, then $f_*(F)$ is injective (use $\mathrm{Hom}(G, f_*(F)) \simeq \mathrm{Hom}(f^{-1}(G), F)$ and the fact that f^{-1} is exact). Similarly, if $F \in \mathbf{Sh}(X)$ is c-soft, then $f_!(F)$ is c-soft. Hence we can apply Proposition 20.8 in both cases. $\square$

Remark 20.10. A particular case of the lemma is given by $Z = \mathrm{pt}$. Then $g_* \simeq \Gamma(Y; -)$ and $\mathrm{R}g_* \simeq \mathrm{R}\Gamma(Y; -)$. We deduce

$$\mathrm{R}\Gamma(X; F) \simeq \mathrm{R}\Gamma(Y; \mathrm{R}f_*(F)) \qquad \text{for any } F \in \mathbf{Sh}(X).$$

Hence $H^i(X; F) \simeq H^i(Y; \mathrm{R}f_*(F)$.

20.1. **Application: example of computation with sheaves.** Let $X = S^{2n-1}$ be the sphere of dimension $2n - 1$. We let $G = \mathbb{Z}/k\mathbb{Z}$ act freely on X (see below). We set $Y = X/G$, $q\colon X \to Y$ the quotient map. We want to compute $H^*(Y; \mathbb{Z}_Y)$. Here is how to build such actions: we consider S^{2n-1} as the unit sphere of $\mathbb{C}^n$ and choose integer $p_1, \dots, p_n$ all primes with k; then the action $[m] \cdot (z_1, \dots, z_n) = (\zeta^{mp_1} z_1, \dots, \zeta^{mp_n} z_n)$, with $\zeta = e^{2i\pi/k}$, is free and preserves S^{2n-1}. The quotient $Y = L_{k,p_1,\dots,p_n}$ is called a lens space. When $n = 2$ we set $L_{p/k} = L_{k,1,p}$. This is a 3 dimensional manifold. The spaces $L_{1/7}$ and $L_{2/7}$ are homotopic but not homeomorpic.

(i) We set $F = q_*(\mathbb{Z}_X)$. We can see that q_* is exact (since q is proper, we have $q_* = q_!$; then we can use Proposition 17.12 to compute the germs of $q_*(F)$ and check that q_* is exact). Hence $q_* \simeq \mathrm{R}q_*$ and by Remark 20.10 we deduce $H^i(Y; F) \simeq H^i(X; \mathbb{Z}_X)$.

(ii) Let us prove that we have an exact sequence

$$0 \to \mathbb{Z}_Y \xrightarrow{a} F \xrightarrow{u} F \xrightarrow{b} \mathbb{Z}_Y \to 0, \tag{20.1}$$

where a is an adjunction morphism corresponding to the adjunction (q^{-1}, q_*), b is a similar morphism and $u = \mathrm{id}_F - \mu_{\mathbf{1}}$ with $\mu_{\mathbf{1}}$ induced by the action of a generator of G. Moreover $b \circ a \colon \mathbb{Z}_Y \to \mathbb{Z}_Y$ is the multiplication by k.

(ii-a) We have $\mathbb{Z}_X = q^{-1}(\mathbb{Z}_Y)$. The adjunction morphism is $a \colon \mathbb{Z}_Y \to q_*q^{-1}(\mathbb{Z}_Y) = F$. We pick $y \in Y$. The fiber $E_y = q^{-1}(y)$ is a set of k points with an action of G and $E_y \simeq G$ as sets with G-action (G acting on itself by addition). Then $(\mathbb{Z}_Y)_y \simeq \mathbb{Z}$, $F_y \simeq \mathbb{Z}^{E_y}$ and a_y is the diagonal map $v \mapsto (v, \dots, v)$.

(ii-b) Let us describe b. For $V \in \mathrm{Op}(Y)$, we have $F(V) = \{f \colon q^{-1}(V) \to \mathbb{Z};\ f$ locally constant$\}$ and $\mathbb{Z}_Y(V) = \{g \colon V \to \mathbb{Z};\ g$ locally constant$\}$. We define $b(V)(f) = g$ where g is the function $g(y) = \sum_{x \in E_y} f(x)$. We can check that g is indeed locally constant, and then that the $b(V)$ give a morphism of sheaves. On the germs the map $b_y \colon F_y \simeq \mathbb{Z}^{E_y} \to (\mathbb{Z}_Y)_y \simeq \mathbb{Z}$ is the sum $b_y(v_1, \dots, v_k) = \sum v_i$.

The description of a, b in the germs give $b \circ a = k\mathrm{id}$.

(ii-c) For a given $g \in G$, g acts on $q^{-1}(V)$ for each $V \in \mathrm{Op}(Y)$. We write $\nu_g \colon q^{-1}(V) \to q^{-1}(V)$ this action. Then we define $\mu_g(V) \colon F(V) \to F(v)$ by $\mu_g(V)(f) = f \circ \nu_{g^{-1}}$. This defines $\mu_g \colon F \to F$. In the germs $(\mu_g)_y$ acts by cyclic permutation on the basis of $F_y = \mathbb{Z}^{E_y}$.

(ii-d) Using the descriptions of a, b, μ in the germs we see that the sequence (20.1) is exact.

(iii) We set $L = \mathrm{coker}(a) = \ker(b) = \mathrm{im}(u)$. We have the short exact sequences $0 \to \mathbb{Z}_Y \xrightarrow{a} F \to L \to 0$, $0 \to L \to F \xrightarrow{b} \mathbb{Z}_Y \to 0$. We have $H^0(Y; F) = H^0(X; \mathbb{Z}_X) = \mathbb{Z}$. We can check $H^0(Y; a) = \mathrm{id}_{\mathbb{Z}}$ and $H^0(Y; b) = k\,\mathrm{id}_{\mathbb{Z}}$. We have $H^i(Y; F') = 0$ for any sheaf F' and $i > \dim Y = 2n - 1$ (see Proposition 21.10 below). Then, by induction on i we find

$$\begin{aligned} H^0(Y; \mathbb{Z}_Y) &= \mathbb{Z}, \\ H^i(Y; \mathbb{Z}_Y) &= \mathbb{Z}/k\mathbb{Z}, \quad \text{for } i \text{ even and } 0 < i < 2n-2, \\ H^{2n-1}(Y; \mathbb{Z}_Y) &= \mathbb{Z}, \\ H^i(Y; \mathbb{Z}_Y) &= 0, \quad \text{else.} \end{aligned}$$

20.2. **Another example.**

Notation 20.11. For a complex $X = (X^{\cdot}, d_X^{\cdot})$ in $\mathbf{C}(\mathcal{C})$ (or in $\mathbf{K}(\mathcal{C})$ or $\mathbf{D}(\mathcal{C})$) and for $k \in \mathbb{Z}$, we denote by $X[k]$ the shifted complex defined by $(X[k])^i = X^{i+k}$ and $d^i_{X[k]} = (-1)^k d^{i+k}_X$.

In particular, for $X \in \mathcal{C}$ viewed as a complex concentrated in degree 0, the complex $X[k]$ is concentrated in degree $-k$.

Definition 20.12. Let $f\colon X \to Y$ be a continuous map of Hausdorff and locally compact spaces. We say that $F \in \mathbf{Sh}(X)$ is f-soft if, for any $y \in Y$, $F|_{f^{-1}(y)}$ is c-soft.

Proposition 20.13. *The family of f-soft sheaves if $f_!$-injective.*

Proposition 20.14. *Let $p\colon \mathbb{R}^{n+d} \to \mathbb{R}^d$ be the projection and let $\mathbf{k}$ be an abelian group. Then $Rp_!(\mathbf{k}_{\mathbb{R}^{n+d}}) \simeq \mathbf{k}_{\mathbb{R}^d}[-n]$. In particular*

$$H^i_c(\mathbb{R}^n; \mathbf{k}_{\mathbb{R}^n}) \simeq \begin{cases} \mathbf{k} & \text{for } i = n, \\ 0 & \text{for } i \neq n. \end{cases}$$

Proof. If $d > 1$ we can write $p = q \circ r$, with $r\colon \mathbb{R}^{n+d} \to \mathbb{R}^{n-1+d}$ and $q\colon \mathbb{R}^{n-1+d} \to \mathbb{R}^d$. By Lemma 20.9 we can prove the result by induction on n, once we have check the case $n = 1$.

In dimension 1 we have seen that (18.1) gives a flabby, hence c-soft, resolution of $\mathbf{k}_{\mathbb{R}}$. On $\mathbb{R}^{d+1}$ we define $\mathcal{F}$ by $\mathcal{F}(U) = \{f\colon U \to \mathbf{k};\ f|_{U \cap (\mathbb{R}^d \times \{y\})}$ is locally constant, for each $y \in \mathbb{R}\}$. We remark that $\mathbf{k}_{\mathbb{R}^{d+1}}$ is a subsheaf of $\mathcal{F}$ and we define $G = \operatorname{coker}(\mathbf{k}_{\mathbb{R}^{d+1}} \to \mathcal{F})$. By definition we have the exact sequence $0 \to \mathbf{k}_{\mathbb{R}^{d+1}} \to \mathcal{F} \to G \to 0$. For each $x \in \mathbb{R}^d$, its restriction to $r^{-1}(x)$ is (18.1), where $r\colon \mathbb{R}^{d+1} \to \mathbb{R}^d$ is the projection. Hence $\mathcal{F}$ and G are r-soft and we can use the resolution to compute $Rr_!(\mathbf{k}_{\mathbb{R}^{d+1}})$. □

We have used a slight generalization of Lemma 18.3 in the proof of Proposition 20.14 and we generalize a bit more in the next lemma. Let $p\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection and, for each $x \in \mathbb{R}$, let $i_x\colon \mathbb{R}^n \times \{x\}$ be the inclusion. For any $F \in \mathbf{Sh}(\mathbb{R}^{n+1})$ we set

$$R_0(F) = \prod_{x \in \mathbb{R}} i_{x*} i_x^{-1}(F).$$

The adjunctions (i_x^{-1}, i_{x*}) give the natural morphisms $F \to i_{x*} i_x^{-1}(F)$. Since $\operatorname{Hom}(F, R_0(F)) \simeq \prod_{x \in \mathbb{R}} i_{x*} \operatorname{Hom}(F, i_x^{-1}(F))$ we obtain a morphism $\varepsilon(F)\colon F \to R_0(F)$. We set $R_1(F) = \operatorname{coker}(\varepsilon(F))$ and get a sequence

$$0 \to F \xrightarrow{\varepsilon(F)} R_0(F) \to R_1(F) \to 0. \tag{20.2}$$

Lemma 20.15. *Let $p\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection and let $F \in \mathbf{Sh}(\mathbb{R}^{n+1})$. Then*

(1) The morphism $\varepsilon(F)$ is a monomorphism and the sequence (20.2) *is exact.*
(2) The sheaves $R_0(F)$ and $R_1(F)$ are p-soft.

Proof. For $y \in \mathbb{R}^n$ we let $j_y\colon \mathbb{R} \to \mathbb{R}^{n+1}$ be the inclusion $x \mapsto (x, y)$. We have $(j_y^{-1}(F))_x \simeq F_{(x,y)}$. Since exactness can be checked at the germs, we can as well restrict first to $\mathbb{R}$ through j_y^{-1}. In the same way, a sheaf G is p-soft if, for each $y \in \mathbb{R}^n$ the sheaf $j_y^{-1}(G)$ is c-soft. Hence we may assume from the beginning that $n = 0$ and we work on $\mathbb{R}$.

(a) For $U \subset \mathbb{R}$ open the morphism $F(U) \to (\prod_{x\in\mathbb{R}}(F_x)_{\{x\}})(U) = \prod_{x\in\mathbb{R}}(F_x)_{\{x\}}(U) = \prod_{x\in U} F_x$ maps a section s to the product of its germs; if the image is 0, then $s_x = 0$ for all $x \in U$ and $s = 0$. Hence $\varepsilon(F)(U)$ is injective.

(b) It is not difficult to see that $R_0(F)$ is flabby, hence c-soft. Let $C \subset \mathbb{R}$ be compact subset and $s \in R_1(F)(C) = \varinjlim_{C\subset U} R_1(F)(U)$, where U is open, be given. We pick U such that s is defined on U.

For each $x \in \overline{C}$ we can choose an open interval $I(x)$ and $t(x) \in R_0(F)(I(x))$ such that $d(t(x)) = s|_{I(x)}$, where $d\colon R_0(F) \to R_1(F)$ is the quotient map. We cover $\overline{C}$ by a finite number of such intervals say $I(x_1), \dots, I(x_N)$. We write $I(x_k) =]a_k, b_k[$. We can assume that the $I(k)$'s are ordered in the sense that $a_k < a_{k+1}$, $b_k < b_{k+1}$. We set $V = \bigcup_{k=1}^N I(x_k)$. We first assume for simplicity that V is connected so $V =]a_1, b_N[$ and we set $W =]a_2, b_{N-1}[$.

Since $R_0(F)$ is c-soft we can find another section $u(x_1) \in R_0(F)(I(x_1))$ which coincides with $t(x_1)$ near $I(x_1) \cap (C \cup W)$ and which is 0 near a_1. We set $s'_1 = d(u(x_1))$. Then $s'_1|_{I(x_1)\cap W}$ coincides with $s|_{I(x_1)\cap W}$. In the same way we can find $s'_N \in R_1(F)(I(x_N))$ such that $s'_N|_{I(x_N)\cap(C\cup W)}$ coincides with $s|_{I(x_N)\cap(C\cup W)}$ and is 0 near b_N. Then s'_1, $s|_W$ and s'_N glue into a section s' of $R_1(F)(V)$ which coincides with s near C and is 0 near a_1 and b_N. Now s' can be extended by 0 on $\mathbb{R}$.

When U has several components we argue in the same way near each component and make the sum of the sections. This gives an extension of s to $\mathbb{R}$ and proves that $R_1(F)$ is c-soft. □

21. More on derived categories

21.1. **Triangulated structure.** Starting with an abelian category $\mathcal{C}$ we have define $\mathbf{C}(\mathcal{C})$ which is abelian, then $\mathbf{K}(\mathcal{C})$ and $\mathbf{D}(\mathcal{C})$. It turns out that the last two categories are not abelian. But they have another structure: they are "triangulated".

Notation 21.1. For a complex $X = (X^{\cdot}, d_X^{\cdot})$ in $\mathbf{C}(\mathcal{C})$ (or in $\mathbf{K}(\mathcal{C})$ or $\mathbf{D}(\mathcal{C})$) and for $k \in \mathbb{Z}$, we denote by $X[k]$ the shifted complex defined by $(X[k])^i = X^{i+k}$ and $d^i_{X[k]} = (-1)^k d_X^{i+k}$.

In particular, for $X \in \mathcal{C}$ viewed as a complex concentrated in degree 0, the complex $X[k]$ is concentrated in degree $-k$.

Some motivation for triangles. We define $K\colon \mathrm{Mor}(\mathcal{C}) \to \mathcal{C}$, $(X \xrightarrow{u} Y) \mapsto \ker(u)$ and $C\colon \mathrm{Mor}(\mathcal{C}) \to \mathcal{C}$, $(X \xrightarrow{u} Y) \mapsto \mathrm{coker}(u)$. We have seen that the functor K is left exact and $\mathrm{R}^0K = K$ (of course), $\mathrm{R}^1K = C$, $\mathrm{R}^iK = 0$, $i \neq 0, 1$. In the same way the functor C is right exact and $\mathrm{L}^0C = C$, $\mathrm{L}^{-1}C = K$, $\mathrm{L}^iC = 0$, $i \neq 0, -1$. In particular $\mathrm{R}^iK = \mathrm{L}^{i-1}C$.

The same computation made in $\mathbf{D}(\mathcal{C})$ shows in fact that $\mathrm{R}K = \mathrm{L}C[-1]$. For a morphism of complexes $u\colon X \to Y$, $\mathrm{L}C(u)$ is called the cone of u, denoted $\mathit{cone}(u)$.

Let $P_1, P_2\colon \mathrm{Mor}(\mathcal{C}) \to \mathcal{C}$ be the source and target, $P_1(X \to Y) = X$, $P_2(X \to Y) = Y$. Then K comes with a morphism $K \to P_1$ and C with a morphism $P_2 \to C$. We deduce $\mathrm{R}K \to P_1$ and $P_2 \to \mathrm{L}C$. Hence, for $u\colon X \to Y$ in $\mathbf{C}(\mathcal{C})$, we have morphisms $Y \to \mathit{cone}(u)$ and $\mathit{cone}(u) \to X[1]$. Summing up we have $X \xrightarrow{u} Y \to \mathit{cone}(u) \to X[1]$ (and we could go on $X[1] \to Y[1] \to \cdots$).

A *triangle* in $\mathbf{D}(\mathcal{C})$ (or $\mathbf{K}(\mathcal{C})$ or $\mathbf{C}(\mathcal{C})$) is the data of three morphisms $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ such that $v \circ u = w \circ v = u[1] \circ w = 0$. A morphism of triangles from $X \to Y \to Z \to X[1]$ to $X' \to Y' \to Z' \to X'[1]$ is the data of three morphisms $X \to X', \dots, Z \to Z'$ making three commutative squares. In $\mathbf{D}(\mathcal{C})$ a triangle is called *distinguished* if it is isomorphic to a cone type triangle as above.

More precise definitions. In fact the cone has an easy description and is well-defined as a functor $\mathit{cone}\colon \mathbf{C}(\mathrm{Mor}(\mathcal{C})) \to \mathbf{C}(\mathcal{C})$ as follows. Remark that $\mathrm{Mor}(\mathbf{C}(\mathcal{C})) \simeq \mathbf{C}(\mathrm{Mor}(\mathcal{C}))$. For $u\colon X \to Y$ we define $\mathit{cone}^i(u) = X^{i+1} \oplus Y^i$ with differential

$$d^i = \begin{pmatrix} -d_X^{i+1} & 0 \\ u^{i+1} & d_Y^i \end{pmatrix}.$$

For $u\colon X \to Y$ we have natural morphisms in $\mathbf{C}(\mathcal{C})$: $\alpha(u)\colon Y \to \mathit{cone}(u)$ and $\beta(u)\colon \mathit{cone}(u) \to X[1]$ (remember the $-$ sign in the differential of $X[1]$). A *mapping cone triangle* in $\mathbf{C}(\mathcal{C})$ is a triangle of the form $X \xrightarrow{u} Y \xrightarrow{\alpha(u)} \mathit{cone}(u) \xrightarrow{\beta(u)} X[1]$.

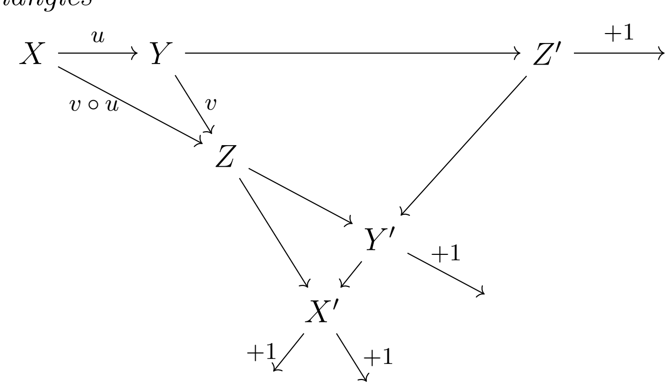
A triangle $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} A[1]$ in $\mathbf{K}(\mathcal{C})$ is called *distinguished* if is isomorphic to the image in $\mathbf{K}(\mathcal{C})$ of a mapping cone triangle of $\mathbf{C}(\mathcal{C})$, $X \xrightarrow{u} Y \xrightarrow{\alpha(u)} \mathit{cone}(u) \xrightarrow{\beta(u)} X[1]$ (which means that we have isomorphisms in $\mathbf{K}(\mathcal{C})$, $f\colon A \to X$, $g\colon B \to Y$, $h\colon C \to \mathit{cone}(u)$, which make commutative squares in $\mathbf{K}(\mathcal{C})$.

Proposition 21.2. *Let $\mathcal{C}$ be an additive category. The distinguished triangles of $\mathbf{K}(\mathcal{C})$ satisfy:*

- *a triangle isomorphic to a distinguished triangle is distinguished,*
- $X \xrightarrow{\mathrm{id}_X} X \to 0 \to X[1]$ *is distinguished,*
- *any $u\colon X \to Y$ can be embedded in a distinguished triangle $X \xrightarrow{u} Y \to Z \to X[1]$,*
- $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ *is distinguished if and only if* $Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$ *is distinguished,*
- *any commutative diagram*

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \vdots & & \downarrow{\scriptstyle f[1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

where the rows are distinguished triangles can be completed into a morphism of triangles,
- octahedron axiom: *any composition $X \xrightarrow{u} Y \xrightarrow{v} Z$ gives rise to a commutative (up to sign) diagram of four distinguished triangles*



(with distinguished triangles really pictured as triangles, the diagram has a octahedron shape).

Definition 21.3. A triangulated category is an additive category endowed with an automorphism, denoted $X \mapsto X[1]$, and a family of triangles, called distinguished, satisfying the axioms of Proposition 21.2.

Proposition 21.4. *Let $\mathcal{C}$ be an abelian category. We say that a triangle in $\mathbf{D}(\mathcal{C})$ is distinguished if it is isomorphic to the image of a distinguished triangle of $\mathbf{K}(\mathcal{C})$ by the localization functor $\mathbf{K}(\mathcal{C}) \to \mathbf{D}(\mathcal{C})$. Then $\mathbf{D}(\mathcal{C})$ is a triangulated category.*

Moreover, if $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ is an exact sequence in $\mathbf{C}(\mathcal{C})$, then there exists a morphism $Z \xrightarrow{w} X[1]$ in $\mathbf{D}(\mathcal{C})$ such that $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is distinguished.

Proposition 21.5. *The derived functors of left or right exact functors send distinguished triangles to distinguished triangles.*

For the next statement we remark that $H^0(X[i]) = H^i(X)$.

Proposition 21.6. *Let $\mathcal{C}$ be an abelian category. The functor $H^0\colon \mathbf{D}(\mathcal{C}) \to \mathcal{C}$ turns distinguished triangles into long exact sequences.*

For a given $X \in \mathbf{D}(\mathcal{C})$, the functors $\mathrm{Hom}_{\mathbf{D}(\mathcal{C})}(X, -)$ and $\mathrm{Hom}_{\mathbf{D}(\mathcal{C})}(-, X)$ also turn distinguished triangles into long exact sequences. This follows from the proposition when we remark that

$$\mathrm{Hom}_{\mathbf{D}(\mathcal{C})}(X, Y[i]) \simeq H^i\mathrm{RHom}(X, Y). \tag{21.1}$$

Indeed both terms are computed by taking an injective resolution of Y (for the LHS by saying that $\mathbf{D}(\mathcal{C})$ is equivalent to $\mathbf{K}(\mathit{injectives})$, for the RHS by the definition of derived functor).

Remark 21.7. We have noticed that $\mathrm{Mor}(\mathbf{C}(\mathcal{C})) \simeq \mathbf{C}(\mathrm{Mor}(\mathcal{C}))$. However it is not true that $\mathrm{Mor}(\mathbf{D}(\mathcal{C})) \simeq \mathbf{D}(\mathrm{Mor}(\mathcal{C}))$ (what about $\mathrm{Mor}(\mathbf{K}(\mathcal{C}))$ and $\mathbf{K}(\mathrm{Mor}(\mathcal{C}))$?). The functor *cone* induces a functor $\mathbf{K}(\mathrm{Mor}(\mathcal{C})) \to \mathbf{K}(\mathcal{C})$. It also sends quasi-isomorphisms to quasi-isomorphisms and induces a functor $\mathbf{D}(\mathrm{Mor}(\mathcal{C})) \to \mathbf{D}(\mathcal{C})$. However, it is not a functor from $\mathrm{Mor}(\mathbf{D}(\mathcal{C}))$ to $\mathbf{D}(\mathcal{C})$. If we have a morphism $v\colon X \to Y$ in $\mathbf{D}(\mathcal{C})$, we can represent v as a morphism of complexes in $\mathbf{C}(\mathcal{C})$, say $v'\colon X' \to Y'$ (for example by taking injective/proejctive resolutions) and compute $\mathit{cone}(v')$; for another representative $v''\colon X'' \to Y''$ we have $\mathit{cone}(v') \simeq \mathit{cone}(v'')$, but not in a canonical way. So, $\mathit{cone}(v)$ is well-defined up to isomorphism, but it is not a functor from $\mathrm{Mor}(\mathbf{D}(\mathcal{C}))$ to $\mathbf{D}(\mathcal{C})$.

21.2. **Resolutions via double complexes.** Let $\mathcal{C}$ be an abelian category and $X = (\cdots \to 0 \to X^i \to X^{i+1} \to \cdots \to X^j \to 0 \to \cdots)$ be an object of $\mathbf{C}(\mathcal{C})$. We assume that we have a commutative diagram

$$\begin{array}{ccccccc}
X^i & \longrightarrow & X^{i+1} & \longrightarrow & \cdots & \longrightarrow & X^j \\
\downarrow & & \downarrow & & & & \downarrow \\
Y^{i,0} & \xrightarrow{d_1^{i,0}} & Y^{i+1,0} & \xrightarrow{d_1^{i+1,0}} & \cdots & \longrightarrow & Y^{j,0} \\
\downarrow{\scriptstyle d_2^{i,0}} & & \downarrow & & & & \downarrow \\
Y^{i,1} & \longrightarrow & Y^{i+1,1} & \longrightarrow & \cdots & \longrightarrow & Y^{j,1} \\
\downarrow{\scriptstyle d_2^{i,1}} & & \downarrow & & & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

where

- the rows $Y^{i,k} \to Y^{i+1,k} \to \cdots \to Y^{j,k}$ are complexes, for each k,
- the columns $X^l \to Y^{l,0} \to Y^{l,1} \to Y^{l,2} \to \cdots$ are resolutions of X^l, for each l.

We define the *total complex* of $Y^{*,*}$ as the complex $\mathrm{Tot}^*(Y)$ where $\mathrm{Tot}^n(Y) = \bigoplus_{p+q=n} Y^{p,q}$ with differential $d^n = \sum_{p+q=n}(d_1^{p,q} + (-1)^p d_2^{p,q})$. Then we can check:

Lemma 21.8. *The morphisms $X^k \to Y^{k,0}$ define a morphism of complexes $X \to \mathrm{Tot}^*(Y)$ which is a quasi-isomorphism.*

Example 21.9. Let $p\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection and $F = (\cdots \to 0 \to F^i \to F^{i+1} \to \cdots \to F^j \to 0 \to \cdots)$ an object of $\mathbf{C}(\mathbf{Sh}(\mathbb{R}^{n+1}))$. We use the functors R_0, R_1 of (20.2) in the double complex

$$\begin{array}{ccccccc}
F^i & \longrightarrow & F^{i+1} & \longrightarrow & \cdots & \longrightarrow & F^j \\
\downarrow & & \downarrow & & & & \downarrow \\
R_0(F^i) & \longrightarrow & R_0(F^{i+1}) & \longrightarrow & \cdots & \longrightarrow & R_0(F^j) \\
\downarrow & & \downarrow & & & & \downarrow \\
R_1(F^i) & \longrightarrow & R_1(F^{i+1}) & \longrightarrow & \cdots & \longrightarrow & R_1(F^j)
\end{array}$$

and we deduce a quasi-isomorphism $F \to G$ where $G = 0 \to R_0(F^i) \to (R_0(F^{i+1}) \oplus R_1(F^i)) \to \cdots \to (R_0(F^j) \oplus R_1(F^{j-1}))R_1(F^j) \to 0$. By Lemma 20.15 G is formed by p-soft sheaves.

Using this example and proceeding as in the proof of Proposition 20.14 we can prove

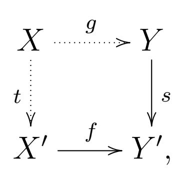
Proposition 21.10. *Let $p\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection and let $F \in \mathbf{C}(\mathbf{Sh}(\mathbb{R}^{n+1}))$ be a complex of sheaves such that $H^iF = 0$ for $i \notin [0, d]$, for some d. Then $H^iRp_!F = 0$ for $i \notin [0, d+1]$. In particular, for any $F \in \mathbf{Sh}(\mathbb{R}^n)$ we have $H^i_c(\mathbb{R}^n; F) \simeq 0$ if $i > n$.*

Proof. Using the truncation functors we can really assume that $F^i = 0$ for $i \notin [0, d]$. Then the complex G found in the example is a p-soft resolution of F of length $d + 1$ and the conclusion follows. □

21.3. **Calculus of fractions.**

Definition 21.11. A family $\mathcal{S}$ of morphisms in $\mathcal{A}$ is a left multiplicative system if

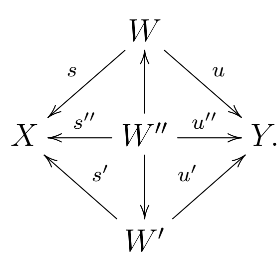
(i) any isomorphism belongs to $\mathcal{S}$,
(ii) if $f, g \in \mathcal{S}$ and $g \circ f$ is defined, then $g \circ f \in \mathcal{S}$,
(iii) for given morphisms f, s, with $s \in \mathcal{S}$, as in the following diagram, there exist g, t, with $t \in \mathcal{S}$, making the diagram commutative



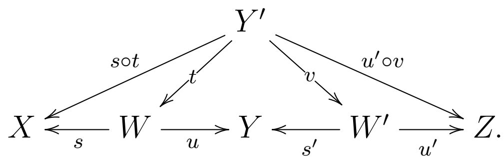
(iv) for two given morphisms $f, g\colon X \to Y$ in $\mathcal{A}$, if there exists $s \in \mathcal{S}$ such that $s \circ f = s \circ g$, then there exists $t \in \mathcal{S}$ such that $f \circ t = g \circ t$:

$$W \overset{t}{\dashrightarrow} X \xrightarrow{f,g} Y \xrightarrow{s} Z.$$

Proposition 21.12. *Let $\mathcal{A}$ be a category and $\mathcal{S}$ a left multiplicative system. Then $\mathcal{A}_{\mathcal{S}}$ can be described as follows. The set of objects is* $\mathrm{Ob}(\mathcal{A}_{\mathcal{S}}) = \mathrm{Ob}(\mathcal{A})$. *For $X, Y \in \mathrm{Ob}(\mathcal{A})$, we have* $\mathrm{Hom}_{\mathcal{A}_{\mathcal{S}}}(X, Y) = \{(W, s, u);\ s\colon W \to X$ *is in $\mathcal{S}$ and $u\colon W \to Y$ is in $\mathcal{A}$*$\}/\sim$, *where the equivalence relation $\sim$ is given by $(W, s, u) \sim (W', s', u')$ if there exists (W'', s'', u''), $s'' \in \mathcal{S}$, such that we have a commutative diagram*



The composition "$u's'^{-1}us^{-1}$" is visualized by the diagram



where t, v, $t \in \mathcal{S}$, are given by (iii) in Definition 21.11.

Let us go back to our abelian category $\mathcal{C}$.

Proposition 21.13. *Let Qis be the family of qis in $\mathbf{K}(\mathcal{C})$. Then Qis is a left (and right) multiplicative system.*

21.4. **Truncation functors.** Let $\mathcal{C}$ be an abelian category. For a given $n \in \mathbb{Z}$ we define $\tau_{\leq n}, \tau_{\geq n}\colon \mathbf{C}(\mathcal{C}) \to \mathbf{C}(\mathcal{C})$ by

$$\tau_{\leq n}(X) = \cdots \to X^{n-2} \to X^{n-1} \to \ker(d_X^n) \to 0 \to \cdots$$
$$\tau_{\geq n}(X) = \cdots \to 0 \to \operatorname{coker}(d_X^{n-1}) \to X^{n+1} \to X^{n+2} \to \cdots .$$

We have natural morphisms in $\mathbf{C}(\mathcal{C})$, for $n \leq m$,

$$\tau_{\leq n}(X) \to X, \qquad X \to \tau_{\geq n}(X),$$
$$\tau_{\leq n}(X) \to \tau_{\leq m}(X), \qquad \tau_{\geq n}(X) \to \tau_{\geq m}(X).$$

We have $H^i(\tau_{\leq n}(X)) \simeq H^i(X)$ for $i \leq n$ and $H^i(\tau_{\leq n}(X)) \simeq 0$ for $i > 0$. We have a similar result for $\tau_{\geq n}(X)$ and the above morphisms induce the tautological morphisms on the cohomology (that is, the identity morphism of H^i if both groups are non-zero, or the zero morphism).

In particular the functors $\tau_{\leq n}$, $\tau_{\geq n}$ send qis to qis and they induce functors, denoted in the same way, on $\mathbf{D}(\mathcal{C})$, together with the same morphisms of functors. We see from the definition, for any $X \in \mathbf{D}(\mathcal{C})$ and any $i \in \mathbb{Z}$:

$$\tau_{\leq i}\tau_{\geq i}(X) \simeq \tau_{\geq i}\tau_{\leq i}(X) \simeq H^i(X)[-i]. \tag{21.2}$$

Lemma 21.14. *Let $\mathcal{C}$ be an abelian category and let $X \in \mathbf{D}(\mathcal{C})$ be an objet concentrated in one degree i_0, that is, $H^i(X) \simeq 0$ if $i \neq i_0$. Then $X \simeq H^{i_0}(X)[-i_0]$.*

Proof. By the hypothesis and by the description of the cohomology of $\tau_{\leq n}(X)$, $\tau_{\geq n}(X)$, the morphisms $\tau_{\leq i_0}(X) \to X$ and $\tau_{\leq i_0}(X) \to \tau_{\geq i_0}(\tau_{\leq i_0}(X))$ are isomorphisms in $\mathbf{D}(\mathcal{C})$. Hence $X \simeq \tau_{\geq i_0}(\tau_{\leq i_0}(X))$ and we conclude with (21.2). □

21.5. **The case of cohomological dimension 1.** The next proposition describes $\mathbf{D}^-(\mathcal{C})$ when $\mathcal{C}$ has *cohomological dimension* 1, which means that $\mathrm{Ext}^i(X,Y) \simeq 0$ for all $i > 1$ and all $X, Y \in \mathcal{C}$. We first give some lemmas.

Lemma 21.15. *Let $\mathcal{C}$ be an abelian category. Then $Q \in \mathcal{C}$ is projective if and only if $\mathrm{Ext}^1(Q,M) \simeq 0$ for all $M \in \mathcal{C}$.*

Proof. The "only if" statement is a particular case of the fact $L^iF(Q) \simeq 0$ for $i > 0$ if Q is projective and F is a right exact functor.

Conversely, let $X \xrightarrow{p} Y \to 0$ be an epimorphism in $\mathcal{C}$. We set $M = \ker(p)$. Hence $0 \to M \to X \to Y \to 0$ is an exact sequence. The long cohomology exact sequence for the functor $\mathrm{Hom}(Q,\cdot)$ is written:

$$\cdots \to \mathrm{Hom}(Q,X) \to \mathrm{Hom}(Q,Y) \to \mathrm{Ext}^1(Q,M) \to \cdots$$

The hypothesis implies that $\mathrm{Hom}(Q,X) \to \mathrm{Hom}(Q,Y)$ is an epimorphism, which proves that Q is projective. □

Lemma 21.16. *Let $\mathcal{C}$ be an abelian category. We assume that for all $X, Y \in \mathcal{C}$ we have $\mathrm{Ext}^2(X,Y) \simeq 0$. Let P be a projective object and let $0 \to Q \xrightarrow{i} P$ be a monomorphism. Then Q is projective.*

Proof. We set $Z = \mathrm{coker}(i)$. Let $M \in \mathcal{C}$ be any object. As in the proof of Lemma 21.15 we have the long exact sequence

$$\cdots \mathrm{Ext}^1(P,M) \to \mathrm{Ext}^1(Q,M) \to \mathrm{Ext}^2(Z,M) \to \cdots$$

Since P is projective, the first term vanishes by Lemma 21.15. The second term vanishes by hypothesis. Hence $\mathrm{Ext}^1(Q,M) \simeq 0$ and Q is projective by Lemma 21.15. □

Exercise 21.17. Let $\mathcal{C}$ be an abelian category with enough projectives such that for all $X, Y \in \mathcal{C}$ we have $\mathrm{Ext}^2(X,Y) \simeq 0$. Prove that $\mathrm{Ext}^i(X,Y) \simeq 0$ for all $i \geq 2$ and all $X, Y \in \mathcal{C}$.

Exercise 21.18. Give a generalization of Lemma 21.16 and Exercise 21.17 with 2 replaced by any $k \geq 2$.

Lemma 21.19. *Let $\mathcal{C}$ be an abelian category and let $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ be an exact sequence in $\mathcal{C}$. We assume that p has a splitting, that is, $j\colon C \to B$ such that $p \circ j = \mathrm{id}_B$. Then i has a splitting, that is, $q\colon B \to A$ such that $q \circ i = \mathrm{id}_A$. Conversely, if i splits, then p splits.*

Proposition 21.20. *Let $\mathcal{C}$ be an abelian category. We assume that for all $X, Y \in \mathcal{C}$ we have $\mathrm{Ext}^2(X,Y) \simeq 0$. We also assume that $\mathcal{C}$ has enough projectives. Then for all $X \in \mathbf{D}^-(\mathcal{C})$ we have $X \simeq \bigoplus_{k\in\mathbb{Z}}(H^k(X))[-k]$. (We remark that $\bigoplus_{k\in\mathbb{Z}}(H^k(X))[-k]$ is the complex L given by $L^k = H^k(X)$ and $d_L^k = 0$ for all $k \in \mathbb{Z}$.)*

Proof. (i) We use the notation L of the proposition. Since $\mathcal{C}$ has enough projectives we can find $P \in \mathbf{D}^-(\mathcal{C})$ such that P^k is projective for all $k \in \mathbb{Z}$ and an isomorphism $X \simeq P$ in $\mathbf{D}^-(\mathcal{C})$.

We will define a morphism $u\colon P \to L$ in $\mathbf{C}(\mathcal{C})$ such that u is a qis. Then u induces the required isomorphism in $\mathbf{D}(\mathcal{C})$. This is the same as giving, for each i, a morphism $u^i\colon P^i \to L^i$ such that $d_P^{i-1} \circ u^i = 0$ and the induced morphism $Z^i(P)/B^i(P) \to L^i$ is an isomorphism.

(ii) We recall that we have monomorphisms $0 \to Z^i(P) \to P^i$ and $0 \to B^i(P) \to Z^i(P)$. By the hypothesis on $\mathcal{C}$ and by Lemma 21.16 we deduce that $Z^i(P)$ and then $B^i(P)$ are projective. By (**??**) we have the exact sequence

$$0 \to Z^i(P) \xrightarrow{a^i} P^i \xrightarrow{b^i} B^{i+1}(P) \to 0. \tag{21.3}$$

Since $B^{i+1}(P)$ is projective, the morphism b^i in (21.3) has a splitting and, by Lemma 21.19, there exists $\alpha^i\colon P^i \to Z^i(P)$ such that $\alpha^i \circ a^i =$ id.

Let $q^i\colon Z^i(P) \to H^i(P) = L^i$ be the natural morphism. We define $u^i\colon P^i \to L^i$ as $u^i = q^i \circ \alpha^i$. Since d^{i-1} factorizes as

$$P^{i-1} \xrightarrow{f^{i-1}} B^i(P) \xrightarrow{g^i} Z^i(P) \xrightarrow{a^i} P^i$$

we have $u^i \circ d^{i-1} = q^i \circ g^i \circ f^{i-1}$ and this vanishes because $q^i \circ g^i = 0$. We see also that the morphism $Z^i(P)/B^i(P) \to L^i$ induced by u^i is the identity morphism of $H^i(P)$. This concludes the proof. □

Example 21.21. We have seen that **Ab** has enough injectives and that an abelian group is injective if and only if it is divisible. It follows easily that a quotient of an injective abelian group is again injective. We deduce that any abelian group M has an injective resolution of length 1: $0 \to M \to I^0 \to I^1 \to 0$. Hence $\operatorname{Ext}^2(N, M) \simeq 0$ for all $M, N \in \mathbf{Ab}$.

21.6. **Example of sheaf computation.** Let X be a Hausdorff and locally compact space, let $Z \subset X$ be a closed subset and $U = X \setminus Z$. We let $j\colon U \to X$, $i\colon Z \to X$ be the inclusions. For $F \in \mathbf{Sh}(X)$ we set

$$F_Z = i_! i^{-1}(F), \qquad F_U = j_! j^{-1}(F).$$

We apply Proposition 17.12 with $f = i$ or $f = j$. Then $f^{-1}(y)$ is empty or a point and $\Gamma_c(f^{-1}(y); F|_{f^{-1}(y)})$ is 0 or F_y. This gives

$$(F_Z)_x = \begin{cases} F_x & \text{if } x \in Z, \\ 0 & \text{if } x \notin Z, \end{cases} \qquad (F_U)_x = \begin{cases} F_x & \text{if } x \in U, \\ 0 & \text{if } x \notin U, \end{cases} \tag{21.4}$$

We remark that i is proper, hence $i_! = i_*$ and $F_Z = i_* i^{-1}(F)$. By the adjunction (i^{-1}, i_*) we have a natural morphism $a\colon F \to F_Z$. By (21.4) we can see that a_x is the identity morphism for $x \in Z$ and $a_x = 0$ for $x \notin Z$. Hence a_x is always surjective and a is an epimorphism.

Lemma 21.22. *We have $F_U|_U \simeq F|_U$ and there exists a unique morphism $b\colon F_U \to F$ such that $b|_U\colon F_U|_U \to F_U$ is the identity morphism.*

Proof. If $V \subset U$ we can see on the definition that $F_U(V) = F(V)$, which proves the first assertion.

Now we pick any open subset $V \subset X$ and $s \in F_U(V)$. By definition $F_U(V) \subset (j_* j^{-1} F)(V) = F(U \cap V)$. The inclusion map $\operatorname{supp}(s) \to V$ is proper. Hence $W = V \setminus \operatorname{supp}(s)$ is open: indeed, for $x \in W$ we choose a compact neighborhood $C \subset V$ of x; then $C \cap \operatorname{supp}(s)$ is compact and $C \setminus (C \cap \operatorname{supp}(s))$ is open and contains x; hence W contains an open neighborhood of any of its point. Of course $s|_{U\cap V\cap W} = 0$. Since F is a sheaf, there exists a unique $\tilde{s} \in F(V)$ such that $\tilde{s}|_{U\cap V} = s|_{U\cap V}$ and $\tilde{s}|_W = 0$.

We define $b(V)\colon F_U(V) \to F(V)$ by $b(V)(s) = \tilde{s}$. When V runs over the open subsets, we can see that gives a sheaf morphism. □

Using (21.4) we see that the following *excision* sequence is exact

$$0 \to F_U \to F \to F_Z \to 0. \tag{21.5}$$

Lemma 21.23. *Let S^n be the sphere of dimension n. Then*

$$H^i(S^n; \mathbf{k}_{S^n}) \simeq \begin{cases} \mathbf{k} & \text{for } i = 0, n, \\ 0 & \text{else.} \end{cases}$$

Proof. We choose a point $x \in S^n$ and set $Z = \{x\}$, $U = S^n \setminus Z$. Let $i\colon Z \to S^n$ $j\colon U \to S^n$ be the inclusions and $a\colon S^n \to \{pt\}$ be the map to the point. We have $a_* = a_!$ since S^n is compact. Then $\Gamma(S^n; F_U) = a_! j_!(j^{-1}(F)) = \Gamma_c(U; j^{-1}(F))$ and $\Gamma(S^n; F_Z) = a_* i_*(i^{-1}(F)) = \Gamma(Z; i^{-1}(F)) = F_x$.

By Proposition 17.12 with $f = i$ or $f = j$ and the fact that $f^{-1}(y)$ is either empty or a point, we see that the functors $i_!$ and $j_!$ are exact. Hence $Ri_! = i_!$, $Rj_! = j_!$. We see also that they send soft sheaves to soft sheaves. Hence $R(a\circ i)_! \simeq Ra_! \circ Ri_!$ and $R(a \circ j)_! \simeq Ra_! \circ Rj_!$. It follows that $\mathrm{R}\Gamma(S^n; F_U) \simeq \mathrm{R}\Gamma_c(U; F|_U)$ and $\mathrm{R}\Gamma(S^n; F_Z) \simeq \mathrm{R}\Gamma(Z; F|_Z)$.

For $F = \mathbf{k}_{S^n}$, the sequence (21.5) becomes $0 \to \mathbf{k}_U \to \mathbf{k}_{S^n} \to \mathbf{k}_Z \to 0$. We deduce the long cohomology sequence $\cdots \to H^i_c(U; \mathbf{k}_U) \to H^i(S^n; \mathbf{k}_{S^n}) \to H^i(Z; \mathbf{k}_Z) \to \cdots$ Since Z is a point we conclude with Proposition 20.14. □

References

[1] M. Kashiwara and P. Schapira, *Sheaves on Manifolds,* Grundlehren der Math. Wiss. **292** Springer-Verlag, (1990).

[2] M. Kashiwara and P. Schapira, *Categories and Sheaves,* Grundlehren der Math. Wiss. **332** Springer-Verlag (2005).

[3] S. Mac Lane, *Categories for the Working Mathematician,* Springer-Verlag, Second Edition (1997).

[4] M.S. Osborne, *Basic homological algebra,* Graduate Texts in Mathematics, **196** Springer-Verlag, New York, (2000).

[5] J. Rotman, *An introduction to homological algebra,* Pure and Applied Mathematics, **85** Academic Press (1979).

[6] Page web de Pierre Schapira

[7] C. Weibel, *An introduction to homological algebra,* Cambridge Studies in Advanced Mathematics, **38** Cambridge University Press, Cambridge, (1994).