HOMOLOGICAL ALGEBRA AND SHEAF THEORY

MASTER 2 LECTURE – NANTES – SEPTEMBER 2022

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Contents

1. Example of derived functors: group (co-)homology

Let G be a group. A (left) G-module is an abelian group A together with a group morphism $\rho\colon G \to \mathrm{Aut}(A)$, where $\mathrm{Aut}(A) = \{\alpha\colon A \to A;$ α is additive and invertible $\}$. We usually forget ρ from the notation and we set, for $g \in G$, $a \in A$, $ga = (\rho(g))(a)$. We can rephrase the definition by "a G-module is an abelian group A with an *action* of G, $G \times A \to A$, $(g, a) \mapsto ga$ such that $e_G a = a$, $g(a + a') = ga + ga'$ and $g(ha) = (gh)a$, for $g, h \in G$, $a, a' \in A$ (and e_G the unit of G). A morphism of G-modules is $u\colon A \to B$ is an additive map such that $u(ga) = gu(a)$ for all $g \in G$, $a \in A$. We let $\mathrm{Hom}_G(A, B)$ be the set of G-module morphisms from A to B. We let $G - \mathrm{Mod}$ be the category of G-modules and morphisms of G-modules.

Let **Ab** be the category of abelian groups and additive morphisms.

1.1. Exact sequences, exact functors.

A composable pair of morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ in $G-\mathrm{Mod}$ is "exact at B" if $\ker g = \mathrm{im}\, f$. A long exact sequence is a sequence $A^n \xrightarrow{d^n} A^{n+1}$, $n \in \mathbb{Z}$, which is exact at each A^n, $n \in \mathbb{Z}$. A short exact sequence is a sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ which is exact at A, B and C, that is, f is injective and $C \simeq B/A$.

A functor $F\colon G - \mathrm{Mod} \to \mathbf{Ab}$ is a "function" on objects and morphisms $A \mapsto F(A)$, $(A \xrightarrow{f} B) \mapsto (F(A) \xrightarrow{F(f)} F(B))$, such that $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for all $A \in G - \mathrm{Mod}$, and $F(g \circ f) = F(g) \circ F(f)$ for all composable morphisms f, g. For the moment we only consider the examples:

1) The functor of *invariants* $A \mapsto A^G = \{a \in A;\ ga = a$ for all $g \in G\}$ and, for $(A \xrightarrow{f} B)$ $f^G = f|_{A^G}$. Note that f^G takes values in B^G because f commutes with the G-action. We remark that A^G is the maximal subgroup of A stable by G and with a trivial action.

2) The functor of *coinvariants* $A \mapsto A_G = A/\mathcal{B}(A)$, where $\mathcal{B}(A)$ is the subgroup of A generated by the elements $ga - a$, $g \in G$, $a \in A$. We remark that, for $f\colon A \to B$, the composition $A \xrightarrow{f} B \to B_G$ sends $\mathcal{B}(A)$ to 0 and defines $f_G\colon A_G \to B_G$. We remark that A^G is the maximal quotient of A with an induced action of G which is trivial.

A functor $F\colon G-\mathrm{Mod} \to \mathbf{Ab}$ is *additive* if the maps $\mathrm{Hom}_G(A, B) \to \mathrm{Hom}(F(A), F(B))$, $f \mapsto F(f)$, are group morphisms, for all $A, B \in G - \mathrm{Mod}$.

An additive functor $F\colon G - \mathrm{Mod} \to G - \mathrm{Mod}$ is *exact* if it sends short exact sequences to short exact sequences. It is *left exact* if, for any exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C$, the sequence $0 \to F(A) \xrightarrow{F(f)}$

$F(B) \xrightarrow{F(g)} F(C)$ is exact. It is *right exact* if, for any exact sequence $A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the sequence $F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \to 0$ is exact.

Lemma 1.1. *The functor* $(-)^G \colon G - \mathrm{Mod} \to \mathbf{Ab}$ *is left exact (but not right exact).*

The functor $(-)^G \colon G - \mathrm{Mod} \to \mathbf{Ab}$ *is right exact (but not left exact).*

1.2. **Projectives, injectives, derived functors.** The starting point of homological algebra is that it makes sense, for a given left (or right) exact functor, to "measure" its deviation from being exact. We will see for example that there exists a first *derived* functor $H^1(G,-)$ such that, for any exact sequence $0 \to A \xrightarrow{u} B \xrightarrow{v} C \to 0$, there exists an exact sequence $0 \to A^G \xrightarrow{u^G} B^G \xrightarrow{v^G} C \to H^1(G,A)$. The fact that $(-)^G$ is not right exact means that $H^1(G,-)$ is not the zero functor.

Definition 1.2. A G-module P is projective if, for any given surjective morphism $v \colon B \to C$ in $G - \mathrm{Mod}$ and any $u \colon P \to C$, there exists $u' \colon P \to B$ such that $u = v \circ u'$:

$$\begin{array}{ccccc} B & \xrightarrow{v} & C & \longrightarrow & 0 \\ {\scriptstyle u'} \uparrow & \nearrow_{u} & & & \\ P & & & & \end{array}$$

We can rephrase the definition by saying that a G-module P is projective if, for any short exact sequence $B \to C \to 0$, the sequence $\mathrm{Hom}_{\mathbf{k}}(P,B) \to \mathrm{Hom}_{\mathbf{k}}(P,C) \to 0$ is exact (it then follows that the functor $\mathrm{Hom}_G(P,\cdot)$ is exact, because exactness of $\mathrm{Hom}_G(P,-)$ at the left is a general fact for any P – check!)

We will see that there exist many projective G-modules and more precisely, for any $A \in G - \mathrm{Mod}$ there exists a projective G-module P and a surjective morphism $P \to A$. We will see that projective modules have a good behaviour with respect to $(-)_G$:

Lemma 1.3. *Let* $0 \to A \xrightarrow{u} B \xrightarrow{v} P \to 0$ *be an exact sequence in* $G - \mathrm{Mod}$. *We assume that* P *is projective. Then the sequence* $0 \to A_G \xrightarrow{u_G} B_G \xrightarrow{v_G} P_G \to 0$ *is exact.*

The idea is to replace an arbitrary G-module by a sequence of projectives.

Definition 1.4. Let $A \in G - \mathrm{Mod}$. A left resolution of A is a long exact sequence

$$\cdots \to P^i \xrightarrow{d^i} P^{i+1} \to \cdots \to P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{\varepsilon} A \to 0.$$

(More precisely, the resolution is $(P^{\cdot}, d^{\cdot})$ and ε is the *augmentation* morphism.) It is called a projective resolution if all the P^i's are projective modules.

Proposition 1.5. *Let $A \in G - \mathrm{Mod}$. Let $(P^{\cdot}, d^{\cdot})$ be a projective left resolution of A and let*

$$\cdots \to P^i_G \xrightarrow{d^i_G} P^{i+1}_G \to \cdots \to P^{-1}_G \xrightarrow{d^{-1}_G} P^0_G \to 0$$

be the sequence obtained by applying $(-)_G$ to this resolution. Then $H_i(G, A) := \ker(d^{-i}_G) / \operatorname{im}(d^{-i-1}_G)$ is independent of the choice of $\{P^{\cdot}, d^{\cdot}\}$.

Then $A \mapsto H_i(G, A)$ is a functor, called the i^{th} *left derived functor* of $(-)_G$. The groups $H_i(G, A)$ are called the homology group of A. We can check that $H_0(G, A) = A_G$.

Proposition 1.6. *Let $0 \to A \xrightarrow{u} B \xrightarrow{v} C \to 0$ be an exact sequence in $G - \mathrm{Mod}$. Then there exists a long exact sequence*

$$H_i(G, A) \xrightarrow{H_i(G,u)} H_i(G, B) \xrightarrow{H_i(G,v)} H_i(G, C) \xrightarrow{\delta^i} H_{i-1}(G, A) \to \cdots \to H_1(G, C) \to A_G \to B_G \to C_G \to 0$$

Reversing the arrows we will also define the notion of *injective* objects and use them to define the *right derived functors* of $(-)^G$, $H^i(G, A)$ (the cohomology of A).

We will give a proof in a more general framework. The groups $H_i(G, A)$, $H^i(G, A)$ are interesting invariants associated with A.

We will introduce similar results for abelian categories. Our main example will be the category of sheaves on a topological space X. Then the derived functors of the "global section" functor will recover the cohomology groups of X, which are the first invariants associated with a manifold.

2. Abelian categories, complexes

2.1. **Categories, the example of sheaves.**

Definition 2.1. A category $\mathcal{C}$ consists of the data of

(i) a set $\mathrm{Ob}(\mathcal{C})$ (the set of *objects*),
(ii) for any $X, Y \in \mathrm{Ob}(\mathcal{C})$, a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ (the set of *morphisms*),
(iii) for any $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a map (the *composition*)

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z), \quad (f, g) \mapsto g \circ f,$$

satisfying

(i) $\circ$ is associative, that is, $(h \circ g) \circ f = h \circ (g \circ f)$ as soon as both sides make sense,
(ii) for each $X \in \mathrm{Ob}(\mathcal{C})$, there exists $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ which is neutral for $\circ$ on the right and on the left, that is, $\mathrm{id}_X \circ f = f$, $g \circ \mathrm{id}_X = g$, as soon as the left hand sides make sense.

We often write $\mathrm{Hom}(X, Y)$ instead of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. We also write $f \colon X \to Y$ instead of $f \in \mathrm{Hom}(X, Y)$. A morphism $f \colon X \to Y$ is an *isomorphism* if there exists $g \colon Y \to X$ (the *inverse* of f) such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. If such a g exists, it is unique.

To avoid logical contradiction we cannot consider the set of all sets. So, when we consider the category **Set** of sets (or the category of groups, rings,...) we assume that we have chosen a set $\mathcal{U}$, called a *universe* which is stable by the operations of set's theory (union, intersection, product,...) and we only consider the categories whose objects and morphisms sets belong to $\mathcal{U}$. We admit that, for any given set X, there exists a universe containing X.

Examples of categories abound (the category of sets, topological spaces, manifolds, rings,...) but we will soon restrict to categories which are similar in some sense to the category of modules over a ring, so called *abelian* categories. It is often possible to associated an abelian category with a priori non linear objects, for example the category of *sheaves of groups* on a topological space is abelian. Since it is a good source of examples we already give the definition.

Definition 2.2. Let X be a topological space. A *presheaf* P of abelian groups on X is the data of

(i) for each open subsets $U \subset X$ an abelian group $P(U)$, the group of *sections over* U,
(ii) for each inclusion of open subsets $V \subset U \subset X$ a morphism of groups $r_{V,U} \colon P(U) \to P(V)$, the *restriction map*, also denoted $s \mapsto s|_V$,

satisfying

(i) $r_{U,U} = \mathrm{id}_{P(U)}$, for each open subsets $U \subset X$,
(ii) $r_{W,V} \circ r_{V,U} = r_{W,U}$, for each inclusion of three open subsets $W \subset V \subset U \subset X$.

A morphism of presheaves $f\colon P \to P'$ is the data of groups morphisms $f(U)\colon P(U) \to P'(U)$ which commute with the restriction maps, that is, $r'_{V,U} \circ f(U) = f(V) \circ r_{V,U}$, for all $V \subset U \subset X$.

Examples 2.3. 1) Let M be an abelian group. The *constant presheaf* of group M on X is the presheaf PA_X defined by $PA_X(U) = M$ for all open subsets $U \subset X$ and $r_{V,U} = \mathrm{id}_M$ for $V \subset U$.

2) We let $\mathcal{C}^0_X(U)$ be the space of continuous functions (with values in $\mathbb{C}$) on an open subset $U \subset X$. Then $U \mapsto \mathcal{C}^0_X(U)$ and the obvious restriction maps define a presheaf $\mathcal{C}^0_M$. When X is a $\mathcal{C}^\infty$-manifold we define in the same way the presheaf of $\mathcal{C}^\infty$-functions, denoted $\mathcal{C}^\infty_X$.

3) Let X be a topological space endowed with a measure μ. We let L^1_X be the presheaf of integrable functions defined by $L^1_X(U) = \{f\colon U \to \mathbb{C};$ f is measurable and $\int_U |f| d\mu < \infty\}$.

Definition 2.4. Let X be a topological space. A *sheaf* F of abelian groups on X is a presheaf satisfying

(i) **separation:** for any open subset $U \subset X$, any open covering $U = \bigcup_{i\in I} U_i$ and any section $s \in F(U)$, if $s|_{U_i} = 0$ for all $i \in I$, then $s = 0$,
(ii) **gluing:** for any open subset $U \subset X$, any open covering $U = \bigcup_{i\in I} U_i$ and any collection of sections $s_i \in F(U_i)$, which are compatible in the sense that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all $i, j \in I$, there exists a section $s \in F(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

A morphism of sheaves $f\colon F \to F'$ is a morphism of the underlying presheaves. We denote by $\mathbf{Psh}(X)$ (resp. $\mathbf{Sh}(X)$) the category of presheaves (resp. sheaves) on X.

Remark 2.5. In Definition 2.2 it is allowed to take the empty family I as the set of indices for a covering. It turns out that it makes sense to ask what it the union of an arbitrary family $\bigcup_{i\in I} X_i$ (below we recall the definition of a coproduct of two objects; we can extend to an arbitrary family) even when $I = \emptyset$: the result is $\bigcup_{i\in\emptyset} X_i = \emptyset$. (What is the product of an empty family of sets, $\prod_{i\in\emptyset} X_i = ?$)

Now we can apply the separation axiom with the covering $\emptyset = \bigcup_{i\in\emptyset} U_i$ of the empty set. Take $s \in F(\emptyset)$. The condition "$s|_{U_i} = 0$ for all $i \in I$" is automatically satisfied since there is nothing to check. So we obtain the conclusion $s = 0$.

(If you don't like the argument, you can add $F(\emptyset) = 0$ in the axioms of sheaves.)

Examples 2.6. The presheaves $\mathcal{C}^0_X$ and $\mathcal{C}^\infty_X$ are sheaves. The presheaves PA_X and L^1_X are not.

Given a presheaf P there exists a "closest possible" sheaf corresponding to P, called the associated sheaf of P and denoted by P^a. We will see a more precise definition when we introduce adjoint functors. For the moment we give an ad hoc definition of P^a.

Definition 2.7. Let X be a topological space and let $P \in \mathbf{Psh}(X)$. For a given point $x \in X$ we set $P_x = \varinjlim_{x \in U} P(U)$, where U runs over the open neighborhoods of x. In other words $P_x = (\bigsqcup_{x \in U} P(U))/\sim$ where $\sim$ is the equivalence relation defined for $s \in P(U)$, $t \in P(V)$ by $s \sim t$ if there exists a third neighborhood of x, $W \subset U \cap V$, such that $s|_W = t|_W$.

The group P_x is called the stalk of P at x. For $s \in P(U)$ its image in P_x is denoted s_x and called the germ of s at x.

For a morphism $u\colon P \to Q$ in $\mathbf{Psh}(X)$ we denote by $u_x\colon P_x \to Q_x$ the induced morphism on the stalks.

Lemma 2.8. *Let F be a sheaf on X and $s \in F(U)$ for some open subset U. Then $s = 0$ if and only if $s_x = 0$ for all $x \in U$.*

Proposition 2.9. *Let $u\colon F \to G$ be a morphism in $\mathbf{Sh}(X)$. Then u is an isomorphism if and only if u_x is an isomorphism for all $x \in X$.*

Proof. See for example [2] Prop. 2.2.2. □

Proposition 2.10. *Let X be a topological space and let $P \in \mathbf{Psh}(X)$. There exist a sheaf P^a and a morphism of presheaves $u\colon P \to P^a$ such that u_x is an isomorphism, for each $x \in X$. Moreover the pair (P^a, u) is unique up to isomorphism.*

Proof. See for example [2] Prop. 2.2.3. We only give a definition of P^a.

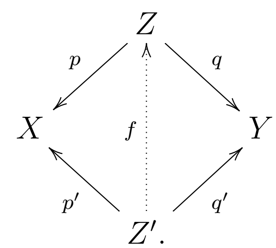
For an open set $U \subset X$ we set $P^a(U) = \{s = (s(x))_{x \in U} \in \prod_{x \in U} P_x;$ for all $x \in U$ there exists a neighborhood V of x in U and $t \in P(V)$ such that $s(y) = t_y$ for all $y \in V\}$. □

Examples 2.11. 1) Let A be an abelian group. The *constant sheaf* of group A on X is the sheaf associated with PA_X, denoted $A_X = (PA_X)^a$. We have $A_X(U) = \{f\colon U \to A;\ f \text{ is locally constant}\}$, where a function f is said locally constant if for any $x \in U$ there exists a neighborhood V of x in U such that $f|_V$ is a constant function. The restriction maps are induced by the inclusions of connected components.

2) Let X be a topological space endowed with a measure μ. Then $(L^1_X)^a = L^{1,loc}_X$ where $L^{1,loc}_X(U) = \{f\colon U \to \mathbb{C};\ f$ is measurable and locally integrable $\}$.

2.2. **Additive categories.**

Definition 2.12. Let $\mathcal{C}$ be a category and $X, Y \in \mathrm{Ob}(\mathcal{C})$. A product of X and Y is an object Z together with morphisms $p\colon Z \to X$, $q\colon Z \to Y$ such that, for any other Z' and $p'\colon Z' \to X$, $q'\colon Z' \to Y$ there exists a unique $f\colon Z' \to Z$ such that $p' = p \circ f$ and $q' = q \circ f$:



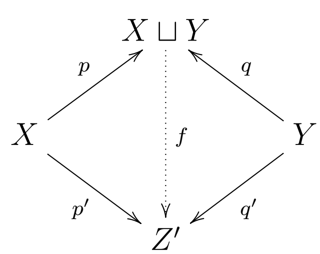
If it exists, the product is unique up to a unique isomorphism. It is denoted $X \times Y$.

We can rephrase the definition of the product by

$$\mathrm{Hom}(Z', X \times Y) = \mathrm{Hom}(Z', X) \times \mathrm{Hom}(Z', Y), \qquad \text{for all } Z' \in \mathrm{Ob}(\mathcal{C}),$$

where the second $\times$ is the product in the category of sets.

A coproduct (sometimes called *sum*) is defined by reversing the arrows. It is denoted $X \sqcup Y$ (or $X \oplus Y$)



We have $\mathrm{Hom}(X \sqcup Y, Z') = \mathrm{Hom}(X, Z') \times \mathrm{Hom}(Y, Z')$.

An object X in a category $\mathcal{C}$ is called *initial* if $\mathrm{Hom}(X, Y)$ consists of a single element for all $Y \in \mathrm{Ob}(\mathcal{C})$. It is called *final* if $\mathrm{Hom}(Y, X)$ consists of a single element for all $Y \in \mathrm{Ob}(\mathcal{C})$. It is called a *zero object* if it is both final and initial. Final, initial or zero objects are unique up to a unique isomorphism, if they exist.

A zero object is usually denoted by 0. If it exists, we also denote by $0 \in \mathrm{Hom}(X, Y)$, for any objects X, Y, the morphism given by the composition $X \to 0 \to Y$. We remark that $0 \circ f = f \circ 0 = 0$ for any f.

Definition 2.13. A category $\mathcal{C}$ is additive if

(i) it has a zero object,
(ii) for any $X, Y \in \mathrm{Ob}(\mathcal{C})$ the product $X \times Y$ and the coproduct $X \sqcup Y$ exist,
(iii) for any $X, Y \in \mathrm{Ob}(\mathcal{C})$ the morphism group $\mathrm{Hom}(X, Y)$ has a structure of abelian group and the composition is bilinear.

We remark that (i) and (ii) are properties whereas (iii) is a priori an additional structure. In fact we can prove that the addition law on $\mathrm{Hom}(X, Y)$ is determined by the properties (i) and (ii) (see for example [3] Thm. 8.2.14).

Let $\mathcal{C}$ be additive and X, Y objects of $\mathcal{C}$. By the definitions of product and coproduct we have natural maps $p_X \colon X \times Y \to X$, $i_X \colon X \to X \sqcup Y$, and similarly p_Y, i_Y. Using the additive structure we define $u = i_X \circ p_X + i_Y \circ p_Y \colon X \times Y \to X \sqcup Y$.

We define $p'_X \colon X \sqcup Y \to X$ corresponding to $(\mathrm{id}_X, 0) \in \mathrm{Hom}(X \sqcup Y, X) \simeq \mathrm{Hom}(X, X) \times \mathrm{Hom}(Y, X)$. We define p'_Y in the same way. Then there exists a unique $v \colon X \sqcup Y \to X \times Y$ corresponding to $(p'_X, p'_Y) \in \mathrm{Hom}(X \sqcup Y, X) \times \mathrm{Hom}(X \sqcup Y, Y)$.

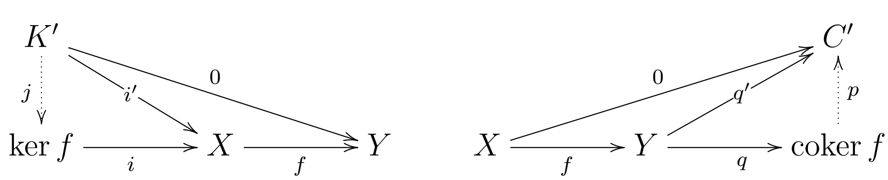
Lemma 2.14. *Let $\mathcal{C}$ be an additive category. Then for any $X, Y \in \mathrm{Ob}(\mathcal{C})$ the natural morphism $u \colon X \times Y \to X \sqcup Y$ is an isomorphism with inverse v.*

This allows us to identify $X \sqcup Y$ and $X \times Y$. In this situation they are usually denoted $X \oplus Y$.

Definition 2.15. Let $\mathcal{C}$ be an additive category and let $f \colon X \to Y$ be a morphism in $\mathcal{C}$. A *kernel* of f is a morphism $i \colon K \to X$ such that $f \circ i = 0$ and such that, for any morphism $i' \colon K' \to X$ satisfying $f \circ i' = 0$ there exists a unique $j \colon K' \to K$ such that $i' = i \circ j$. If the kernel exists, it is unique up to a unique isomorphism and we set $\ker f = K$.

A *cokernel* of f is a kernel in the opposite category. It is denoted $\mathrm{coker}\, f$.

This is visualized by the diagrams:



We can also rephrase the definitions by, for all $Z \in \mathrm{Ob}(\mathcal{C})$:

$$\mathrm{Hom}(Z, \ker(f)) \simeq \ker(\varphi_f \colon \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z, Y)),$$

$$\mathrm{Hom}(\mathrm{coker}(f), Z) \simeq \ker(\psi_f \colon \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)),$$

with $\varphi_f(u) = f \circ u$, $\psi_f(u) = u \circ f$.

2.3. Exercises.

Exercise 2.16. Inductive limit (also called "colimit"). We give a definition in the special case of a filtrant indexing set. Let $(\mathcal{I}, \leq)$ be an ordered set which is *filtrant*, which means: for any $i, j \in \mathcal{I}$ there exists $k \in \mathcal{I}$ such that $i \leq k$ and $j \leq k$. Typical examples are $\mathcal{I} = \mathbb{N}$ and, for a topological space X and a point $x \in X$, $\mathcal{I}$ is the set of open neighborhoods of x.

Let $\{E_i, u_{ji}\}$ be an inductive system of sets indexed by $\mathcal{I}$, which means: u_{ji} is a map $u_{ji}\colon E_i \to E_j$ for any $i \leq j$ such that $u_{ii} = \mathrm{id}_{E_i}$ and $u_{kj} \circ u_{ji} = u_{ki}$ when $i \leq j \leq k$. Then

$$\varinjlim_{i \in \mathcal{I}} E_i = \bigsqcup_{i \in \mathcal{I}} E_i / \sim,$$

where $\sim$ is the equivalence relation defined by $x_i \in E_i \ \sim \ x_j \in E_j$ if there exists k with $i, j \leq k$ and $u_{ki}(x_i) = u_{kj}(x_j)$. This set comes with natural maps $u_i\colon E_i \to \varinjlim_{i \in \mathcal{I}} E_i$ induced by the inclusion of E_i in $\bigsqcup E_k$. We remark that any element of $\varinjlim_{i \in \mathcal{I}} E_i$ is represented by an element $x_{i_0} \in E_{i_0}$ for some $i_0 \in \mathcal{I}$.

(1) Check that, if the E_i are groups and the u_{ji} are group morphisms, then $\varinjlim_{i \in \mathcal{I}} E_i$ has a unique group structure such that the maps u_i are group morphisms.

(2) When $\mathcal{I} = \mathbb{N}$ we only need to specify the maps $u_{i+1,i}$. Take $E_i = \mathbb{Z}$ for all i and $u_{i+1,i}(x) = 2x$ for all i. We write for short $\varinjlim_{i \in \mathbb{N}} E_i = \varinjlim(\mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \xrightarrow{2\cdot} \cdots)$. What is this colimit ? (Identify with a subgroup of $\mathbb{Q}$.)

(3) Give an example of an inductive system of groups indexed by $\mathbb{N}$, $E_0 \xrightarrow{u_{1,0}} E_1 \to \cdots$, where all groups and all maps $u_{i+1,i}$ are non zero, but $\varinjlim_{i \in \mathbb{N}} E_i \simeq 0$.

(4) Let P be a presheaf on a topological space X and $x \in X$. We assume that x has a countable system of decreasing open neighborhoods B_n. (Typically $X = \mathbb{R}^n$ and B_n is the open ball with center x and radius $1/n$.) Check that $P_x \simeq \varinjlim_{n \in \mathbb{N}} P(B_n)$.

Exercise 2.17. Let X be a topological space, F a sheaf on X and $s \in F(U)$ for some open subset U. We assume that $s_x = 0$ for some $x \in U$. Prove that there exists an open neighborhood $V \subset U$ of x such that $s|_V = 0$. Prove Lemma 2.8.

Prove Proposition 2.9: the difficult part is the reverse direction. Check first that, for an open set U, the map $F(U) \to G(U)$ is injective, using Lemma 2.8. For the surjectivity, we pick $s \in G(U)$. For a given $x \in U$ we can find $t \in F_x$ which is mapped to s_x. This t is represented by some section $\tilde{t} \in F(V)$. First prove that $(\tilde{t}|_W) = s|_W$ for some small enough neighborhood W of x.

Exercise 2.18. Skyscraper sheaf. Let X be a separated topological space, $x \in X$ and A an abelian group. Prove that there exists a unique sheaf, denoted $A_{\{x\}}$, such that $(A_{\{x\}})_y \simeq 0$ for all $y \neq x$ and $(A_{\{x\}})_x \simeq A$. Describe the sections $A_{\{x\}}(U)$.

Let F be a sheaf on X. Remark that there is a natural morphism $i_x \colon F \to (F_x)_{\{x\}}$ which induces the identity morphism on the stalk at x.

We define the product $\widetilde{F} = \prod_{x \in X} (F_x)_{\{x\}}$ by $\widetilde{F}(U) = \prod_{x \in X}((F_x)_{\{x\}}(U))$. In fact we have $\widetilde{F}(U) = \prod_{x \in U} F_x$ (which makes sense even if X is not separated). The morphisms i_x together, for all $x \in X$, give $F \to \widetilde{F}$. Prove that F is a subsheaf of $\widetilde{F}$ in the sense that the map $F(U) \to \widetilde{F}(U)$ is injective, for all open sets U.

Exercise 2.19. Let P be a presheaf. We assume that there exists a sheaf F and a morphism $u \colon P \to F$ such that the induced morphisms on the stalks $u_x \colon P_x \to F_x$ are all isomorphisms (as in Proposition 2.10). Define $\widetilde{F}$ as in Exercise 2.18. Deduce from this exercise that $F(U)$ is a subgroup of $\prod_{x \in U} P_x$, for any open subset U. Then check that $F(U)$ must be as described in the proof of Proposition 2.10.

Exercise 2.20. A morphism $f \colon X \to Y$ in a category $\mathcal{C}$ is called a *monomorphism* if, for all $W \in \mathrm{Ob}(\mathcal{C})$ and all morphisms $g, h \colon W \to X$ in $\mathcal{C}$, the equality $f \circ g = f \circ h$ implies $g = h$ (in other words, the map $\mathrm{Hom}_{\mathcal{C}}(W, X) \to \mathrm{Hom}_{\mathcal{C}}(W, Y)$, $g \mapsto f \circ g$, is injective).

If $\mathcal{C}$ is an additive category, prove that f is an monomorphism if and only if the kernel of f exists and is 0.

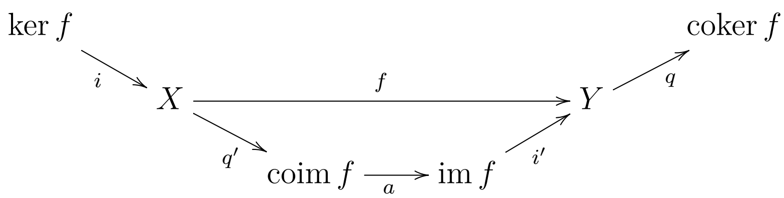
Define the dual notion of *epimorphism* and prove that f is an epimorphism if and only if $\operatorname{coker} f \simeq 0$.

Exercise 2.21. Let $\mathcal{C}$ be an additive category and let $f \colon X \to Y$ be a morphism in $\mathcal{C}$. We assume that $\ker f$ exists. Prove that the morphism $i \colon \ker f \to X$ is a monomorphism.

Dually, if f has a cokernel, $Y \to \operatorname{coker} f$ is an epimorphism.

2.4. **Abelian categories.**

Lemma 2.22. *Let $\mathcal{C}$ be an additive category and let $f\colon X \to Y$ be a morphism which admits a kernel* $\ker f \xrightarrow{i} X$ *and a cokernel* $Y \xrightarrow{q} \operatorname{coker} f$. *We also assume that i has a cokernel (it is called the* coimage *of f, denoted $X \xrightarrow{q'} \operatorname{coim} f$) and that q has a kernel (it is called the* image *of f, denoted* $\operatorname{im} f \xrightarrow{i'} Y$*). Then there exists a unique morphism $a\colon \operatorname{coim} f \to \operatorname{im} f$ such that $f = i' \circ a \circ q'$.*



Proof. The existence of a follows from the universal properties of ker and coker. If we have another a' making the diagram commute, then $a \circ q' = a' \circ q'$ because i' is a monomorphism. And then $a = a'$ because q' is an epimorphism. □

Definition 2.23. An *abelian* category $\mathcal{C}$ is an additive category such that, for any morphism $f\colon X \to Y$, the kernel and the cokernel of f exist (hence also the image and the coimage) and the natural morphism $\operatorname{coim} f \to \operatorname{im} f$ of Lemma 2.22 is an isomorphism.

The typical example is the category **Ab** of abelian groups. For a group G the category $G - \mathrm{Mod}$ is abelian.

Example 2.24. Let $\mathbf{k}$ be a field and $\mathbf{Vect}_{fil}$ be the category of filtered vector spaces over $\mathbf{k}$. The objects, denoted $(V, F^{\cdot})$, are vector spaces V together with sequences of subspaces $\cdots F^iV \subset F^{i+1}V \subset F^{i+2}V \cdots \subset V$ where $i \in \mathbb{Z}$ such that $V = \bigcup_{i\in\mathbb{N}} V_i$. The morphisms from $(V, F^{\cdot})$ to $(W, F^{\cdot})$ are linear maps $u\colon V \to W$ such that $u(F^iV) \subset F^iW$.

The category $\mathbf{Vect}_{fil}$ is an additive category with kernels and cokernels. For $u\colon (V, F^{\cdot}) \to (W, F^{\cdot})$, we can check that $\ker u$ is the usual $\ker u$ with the filtration $F^i(\ker u) = \ker(u|_{F^iV})$ and $\operatorname{coker} u$ is the usual $\operatorname{coker} u$ with the filtration $F^i(\operatorname{coker} u) = F^iW/(F^iW \cap \operatorname{im} u)$.

We set $V = \mathbf{k}$ with two filtrations $F_1^iV = 0$ for $i \le 0$, $F_1^iV = \mathbf{k}$ for $i > 0$ and $F_2^iV = F_1^{i+1}V$. The identity map on V induces a morphism $u\colon (V, F_1^{\cdot}) \to (V, F_2^{\cdot})$. Then $\operatorname{coim}(u) = (V, F_1^{\cdot}) \not\simeq \operatorname{im}(u) = (V, F_2^{\cdot})$.

Lemma 2.25. *Let $f\colon X \to Y$ be a morphism in an abelian category. Then f is an isomorphism if and only if* $\ker f \simeq 0$ *and* $\operatorname{coker} f \simeq 0$.

Notation 2.26. Let $f\colon X \to Y$ be a morphism in an abelian category such that $\ker f \simeq 0$. We often write $Y/X := \operatorname{coker} f$.

Lemma 2.27. *Let $\mathcal{C}$ be an abelian category and let*

$$\begin{array}{ccc} X & \xrightarrow{u} & X' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{v} & Y' \end{array}$$

be a commutative diagram in $\mathcal{C}$. Then there exist unique morphisms $\ker u \to \ker v$ *and* $\operatorname{coker} u \to \operatorname{coker} v$ *such that the following diagram commutes*

$$\begin{array}{ccccccc} \ker u & \longrightarrow & X & \xrightarrow{u} & X' & \longrightarrow & \operatorname{coker} u \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \downarrow \\ \ker v & \longrightarrow & Y & \xrightarrow{v} & Y' & \longrightarrow & \operatorname{coker} v. \end{array}$$

Let X be a topological space.

Proposition 2.28. *The category* $\mathbf{Psh}(X)$ *is abelian. Moreover, for* $u\colon P \to P'$ *in* $\mathbf{Psh}(X)$*, we have, for any open subset* $U \subset X$*,*

$$(\ker u)(U) \simeq \ker(u(U)\colon P(U) \to P'(U)),$$
$$(\operatorname{coker} u)(U) \simeq \operatorname{coker}(u(U)\colon P(U) \to P'(U)),$$

and, for all $x \in X$*,* $(\ker u)_x \simeq \ker(u_x)$ *and* $(\operatorname{coker} u)_x \simeq \operatorname{coker}(u_x)$*.*

Remark 2.29. Let $\mathcal{C}$ be a category and $\mathcal{C}'$ a full subcategory of $\mathcal{C}$, which means that $\mathrm{Ob}(\mathcal{C}')$ is a subset of $\mathrm{Ob}(\mathcal{C})$ and that $\mathrm{Hom}_{\mathcal{C}'}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$ for any $X, Y \in \mathcal{C}'$.

For $X, Y \in \mathcal{C}'$, if the product $X \times Y$ exists in $\mathcal{C}$ and belongs to $\mathcal{C}'$, then it is the product of X and Y in $\mathcal{C}'$. A similar remark holds for the coproduct, the kernel and the cokernel.

Proposition 2.30. *The category* $\mathbf{Sh}(X)$ *is abelian. Moreover, for* $u\colon F \to F'$ *in* $\mathbf{Sh}(X)$*, we have, denoting by* $\bar{u}$ *the morphism* u *viewed in* $\mathbf{Psh}(X)$*,*

(a) $\ker \bar{u}$ *is a sheaf and* $\ker u \simeq \ker \bar{u}$*,*
(b) $\operatorname{coker} u \simeq (\operatorname{coker} \bar{u})^a$*,*
(c) *for all* $x \in X$*,* $(\ker u)_x \simeq \ker(u_x)$ *and* $(\operatorname{coker} u)_x \simeq \operatorname{coker}(u_x)$*.*

Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence of morphisms in an abelian category. We assume that $g \circ f = 0$. Then there exists a natural morphism $a\colon \operatorname{im} f \to \ker g$. We have $\ker a \simeq 0$. We say that the sequence is exact (at Y) if this morphism is an isomorphism, that is, $\operatorname{coker} a \simeq 0$. In general we set

$$H(X \xrightarrow{f} Y \xrightarrow{g} Z) := \operatorname{coker}(\operatorname{im} f \to \ker g)$$

and call this object the *cohomology* of the sequence. A short sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is said exact if it is exact at X, Y and Z.

Lemma 2.31. *Let X be a topological space. If a sequence $F \xrightarrow{u} G \xrightarrow{v} H$ is exact in $\mathbf{Psh}(X)$, then the sequence $F_x \xrightarrow{u_x} G_x \xrightarrow{v_x} H_x$ is exact for each $x \in X$.*

A sequence $F \xrightarrow{u} G \xrightarrow{v} H$ in $\mathbf{Sh}(X)$ is exact if and only if the sequence $F_x \xrightarrow{u_x} G_x \xrightarrow{v_x} H_x$ is exact for each $x \in X$.

2.5. **Exercises.**

Exercise 2.32. Let X be a topological space, and $Z \subset X$ a closed subset. Let A be an abelian group. We define the presheaf $PA_{X,Z}$ on X by $PA_{X,Z}(U) = 0$ if $Z \cap U = \emptyset$ and $PA_{X,Z}(U) = A$ if $Z \cap U \neq \emptyset$ with the restriction maps $r_{V,U} = \mathrm{id}_A$ if $V \cap Z \neq \emptyset$ (otherwise $r_{V,U}$ must be 0).

We set $A_{X,Z} = (PA_{X,Z})^a$. Using the construction of P^a in the proof of Proposition 2.10 check that $A_{X,Z}(U) = \{f : U \cap Z \to A;\ f \text{ is locally constant}\}$. (Here, locally constant means: any $x \in U \cap Z$ has a neighborhood $V(x)$ such that $f|_{V(x)}$ is constant.) In particular, if X is locally connected, then $A_{X,Z}(U) \simeq A^{\pi_0(Z \cap U)}$, where $\pi_0(Z \cap U)$ is the set of connected components of $Z \cap U$. Note that, when X is not locally connected, for example $X = \mathbb{Q}$, this last description does not hold.

Check that $(A_{X,Z})_x \simeq A$ if $x \in Z$ and $(A_{X,Z})_x \simeq 0$ otherwise.

Exercise 2.33. In the previous exercise we could try to remove the condition "Z is closed" but the result is more complicated. Set $A'_{X,Z} = (PA_{X,Z})^a$ for a general Z. Describe $A'_{X,Z}$ if we take (1) $X = \mathbb{R}^n$, Z an open ball in $\mathbb{R}^n$ (2) $X = \mathbb{R}$, $Z = \mathbb{R} \setminus \{0\}$. In particular the stalks $(A'_{X,Z})_x$ don't satisfy the same relation as in the previous exercise.

Beware that we will use the notation $A_{X,Z}$ for more general Z but only the case Z closed is given by the above construction.

Exercise 2.34. Let $\mathcal{C}$ be an abelian category and let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be two morphisms in $\mathcal{C}$. We assume that $\ker(g \circ f) = 0$. Prove that $\ker(f) = 0$.

Exercise 2.35. Prove Lemma 2.27.

Let $\mathcal{C}$ be an abelian category. We consider the commutative diagram in $\mathcal{C}$

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} \\ 0 & \longrightarrow & X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' \end{array}$$

and we assume that the rows are exact (at X, Y and X', Y'). By Lemma 2.27 this commutative diagram gives two morphisms $\ker u \xrightarrow{f_0} \ker v \xrightarrow{g_0} \ker w$. Prove that this sequence is exact at $\ker v$.

Prove that the same is true when we start from

$$\begin{array}{ccccccc} & & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} \\ 0 & \longrightarrow & X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' \end{array}$$

where the first row is only exact at Y and the second only at X'.

Exercise 2.36. Let $\mathcal{C}$ be a category. We define the category of morphisms in $\mathcal{C}$, say $\mathrm{Mor}(\mathcal{C})$, as the category whose objects are the morphisms in $\mathcal{C}$ (that is an object is the data of $X \xrightarrow{u} X'$) and the morphisms are the commutative diagrams $\mathrm{Hom}_{\mathrm{Mor}(\mathcal{C})}((X \xrightarrow{u} X'), (Y \xrightarrow{v} Y')) = \{(f, f'); f\colon X \to Y, f'\colon X' \to Y', v \circ f = f' \circ u\}$. The composition is given termwise by the composition in $\mathcal{C}$, that is, $(g, g') \circ (f, f') = (g \circ f, g' \circ f')$.

We assume that $\mathcal{C}$ is abelian. Prove that $\mathrm{Mor}(\mathcal{C})$ is also abelian. (To save time, you can admit that $\mathrm{Mor}(\mathcal{C})$ is additive and check only the existence of kernels, cokernels and Definition 2.23 using Lemma 2.27.)

Exercise 2.37. We let $\mathcal{O}_{\mathbb{C}} \in \mathbf{Sh}(\mathbb{C})$ be the sheaf of holomorphic functions over $\mathbb{C}$, that is, $\mathcal{O}_{\mathbb{C}}(U) = \{f\colon U \to \mathbb{C};\ f \text{ is holomorphic}\}$. We let $\mathcal{O}_{\mathbb{C}}^{\times}$ be the sheaf of non vanishing holomorphic functions and we denote by $\exp\colon \mathcal{O}_{\mathbb{C}} \to \mathcal{O}_{\mathbb{C}}^{\times}$ the morphism $f \mapsto \exp(f)$. Prove that we have an exact sequence $0 \to \mathbb{Z}_{\mathbb{C}} \to \mathcal{O}_{\mathbb{C}} \xrightarrow{\exp} \mathcal{O}_{\mathbb{C}}^{\times} \to 0$ in $\mathbf{Sh}(\mathbb{C})$.

Prove that this sequence is not exact in $\mathbf{Psh}(\mathbb{C})$.

Exercise 2.38. (Variation on Exercise 2.37) We keep the notations of Exercise 2.37 Let $u\colon \mathcal{O}_{\mathbb{C}} \to \mathcal{O}_{\mathbb{C}}$ be the derivation, that is, $u(U)(f) = f'$ for $f \in \mathcal{O}_{\mathbb{C}}(U)$. What is $\ker u$, $\operatorname{coker} u$ in $\mathbf{Sh}(\mathbb{C})$? Prove that u is not surjective in $\mathbf{Psh}(\mathbb{C})$.

Exercise 2.39. We keep the framework of Exercise 2.16. Let $\mathcal{I}$ be a filtrant ordered set. Let $\{E_i, u_{ji}\}$, $\{F_i, v_{ji}\}$ be inductive systems indexed by $\mathcal{I}$. We remark that $\varinjlim_{i\in\mathcal{I}} E_i$ comes with maps $\pi_i\colon E_i \to \varinjlim_{i\in\mathcal{I}} E_i$ (we use abusively the same notation for the F_i's). We assume to be given maps $f_i\colon E_i \to F_i$ commuting with the u_{ji}, v_{ji}.

Check that these maps induce a unique map $f\colon \varinjlim_{i\in\mathcal{I}} E_i \to \varinjlim_{i\in\mathcal{I}} F_i$ such that $f \circ \pi_i = \pi_i \circ f_i$ for all i.

Now we assume that our inductive systems are made of abelian groups and all maps are additive. Then $\varinjlim_{i\in\mathcal{I}} E_i$ is an abelian group. We remark that u_{ji} maps $\ker(f_i)$ to $\ker(f_j)$ and we obtain an inductive system $\{\ker(f_i), u_{ji}\}$. Check that $\varinjlim_{i\in\mathcal{I}} \ker(f_i) \simeq \ker(f)$.

In the same way check that $\varinjlim_{i\in\mathcal{I}} \operatorname{coker}(f_i) \simeq \operatorname{coker}(f)$.

Exercise 2.40. Let X be a topological space and $Z \subset X$ a closed subset. Let A be an abelian group. Recall the "constant sheaf on Z" $A_{X,Z}$, such that $A_{X,Z}(U) = \{f\colon U \cap Z \to A;\ f \text{ is locally constant}\}$.

Let $Z' \subset Z$ be a closed subset. We define a restriction morphism $r\colon A_{X,Z} \to A_{X,Z'}$ by $r(U)(f) = f|_{U\cap Z'}$ (check that this is a sheaf morphism). For an open subset $V \subset X$ we define

$$A_{X,U} = \ker(A_{X,X} \to A_{X,X\setminus U}).$$

Verify that $(A_{X,U})_x \simeq \begin{cases} A & \text{if } x \in U, \\ 0 & \text{if } x \notin U. \end{cases}$

We set $F = A_{X,U} \oplus A_{X,X\setminus U}$. Check that $F_x \simeq (A_{X,X})_x$ for all $x \in X$.

We consider $X = \mathbb{R}$, $U =]-\infty, 0[$ and $Z = [0, +\infty[$. Prove that $\mathrm{Hom}(A_{X,Z}, A_{X,X}) = 0$.

With F as above, prove that $F \not\simeq A_{X,X}$.

3. Categories of complexes

Definition 3.1. Let $\mathcal{C}$ be an additive category. A complex $(X^\cdot, d_X^\cdot)$ in $\mathcal{C}$ is a sequence of composable morphisms in $\mathcal{C}$

$$\cdots \to X^i \xrightarrow{d_X^i} X^{i+1} \to \cdots$$

such that $d^{i+1} \circ d^i = 0$, for all $i \in \mathbb{Z}$ (we forget the subscripts when there is no ambiguity). The morphisms d_X^i are called the differentials.

A morphism f from a complex $(X^\cdot, d_X^\cdot)$ to a complex $(Y^\cdot, d_Y^\cdot)$ is a sequence of morphisms $f^i \colon X^i \to Y^i$, $i \in \mathbb{Z}$, commuting with the differentials.

We denote by $\mathbf{C}(\mathcal{C})$ the category of complexes in $\mathcal{C}$. A complex is said bounded from below (resp. above) if $X^i \simeq 0$ for $i \ll 0$ (resp. $i \gg 0$). It is bounded if it is bounded from below and from above. We let $\mathbf{C}^+(\mathcal{C})$, $\mathbf{C}^-(\mathcal{C})$, $\mathbf{C}^b(\mathcal{C})$ be the corresponding categories.

We collect some facts about abelian categories. Any morphism $f \colon A \to B$ in an abelian category factorizes as $f = i_f \circ p_f$, with $p_f \colon A \to \operatorname{im} f$, $i_f \colon \operatorname{im} f \to B$ (see the diagram in Lemma 2.22 and the fact that a in this diagram is an isomorphism by definition). Moreover p_f is an epimorphism and i_f a monomorphism (see Exercises 2.20 and 2.21). Since i_f is a monomorphism, a morphism $x \colon X \to A$ satisfies $p_f \circ x = 0$ if and only if $f \circ x = 0$. It follows that $\ker f \simeq \ker p_f$. Hence we have an exact sequence (as in **Ab**)

$$0 \to \ker f \to A \to \operatorname{im} f \to 0. \tag{3.1}$$

In the same way $\operatorname{coker} i_f \simeq \operatorname{coker} f$ and we have an exact sequence

$$0 \to \operatorname{im} f \to B \to \operatorname{coker} f \to 0. \tag{3.2}$$

Now let $g \colon B \to C$ be another morphism such that $g \circ f = 0$. Let $j_g \colon \ker g \to B$ be the natural morphism. Then $(g \circ i_f) \circ p_f = 0$ and, since p_f is an epimorphism, $g \circ i_f = 0$. We deduce a uniquely defined morphism $j \colon \operatorname{im} f \to \ker g$ such that $i_f = j_g \circ j$. Since i_f is a monomorphism, we see that j is also a monomorphism (check!). We write $\operatorname{im} f \hookrightarrow \ker g \hookrightarrow B$ and, since j is a monomorphism, we often use the notation $\operatorname{coker} j = \ker g / \operatorname{im} f$.

Definition 3.2. Let $\mathcal{C}$ be an abelian category and let $X = (X^\cdot, d_X^\cdot) \in \mathbf{C}(\mathcal{C})$. For $i \in \mathbb{Z}$ we define

$$Z^i(X) = \ker d_X^i, \qquad B^i(X) = \operatorname{im} d_X^{i-1},$$
$$H^i(X) = Z^i(X)/B^i(X) = \operatorname{coker}(B^i(X) \to Z^i(X))$$

and we call $H^i(X)$ the i^{th} cohomology of X. In the case of the category of groups $Z^i(X)$ (resp. $B^i(X)$) is called the i^{th} group of cocycles (resp. boundaries).

For a morphisms of complexes $f\colon X \to Y$ we denote by $Z^i(f)$, $B^i(f)$, $H^i(f)$ the induced morphisms, which exist and are well-defined by Lemma 2.27. Hence Z^i, B^i, H^i are functors from $\mathbf{C}(\mathcal{C})$ to $\mathcal{C}$ (see Definition 4.1).

If $\mathcal{C}$ is abelian, then $\mathbf{C}(\mathcal{C})$ is also abelian. Moreover for a morphism $f\colon X \to Y$ in $\mathbf{C}(\mathcal{C})$ the kernel satisfies $(\ker f)^i = \ker(f^i)$ and the differential $d^i_{\ker f}$ is the natural morphism $\ker(f^i) \to \ker(f^{i+1})$ given by Lemma 2.27. The same holds for the cokernel.

We will use several preliminary lemmas, which are not too difficult to prove in $\mathbf{Ab}$. We only indicate once the proof for a general abelian category. For this the following notion is useful.

Definition 3.3. Let $u\colon A \to B$, $f\colon B' \to B$ be morphisms in an abelian category $\mathcal{C}$. We set $A \times_B B' = \ker(s\colon A \oplus B' \to B)$, where $s = (u, -f)$. It comes with a commutative square

$$\begin{array}{ccc} A \times_B B' & \xrightarrow{u'} & B' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{u} & B \end{array}$$

and is universal for this property (any C with maps to A and B' making a commutative square factorizes uniquely though $A \times_B B'$). It is called the fiber product of A and B' over B.

When $\mathcal{C} = \mathrm{Mod}(\mathbf{k})$ we have $A \times_B B' = \{(a, b') \in A \oplus B';\ u(a) = f(b')\}$.

Let us define $i\colon \ker u \to A \oplus B'$ by $i = (j_u, 0)$ where j_u is the morphism $\ker u \to A$. Then $s \circ i = 0$ and i factorizes through a uniquely defined morphism $i'\colon \ker u \to A \times_B B'$. If $q\colon A \oplus B' \to B'$ denotes the natural map, we have $q \circ i = 0$, hence $u' \circ i' = 0$ and i' induces a morphism

$$\ker u \to \ker u'. \tag{3.3}$$

Since a morphism $x\colon X \to A$ such that $u \circ x = 0$ factorizes uniquely through $\ker u$, we deduce that x factorizes through $\tilde{x}\colon X \to A \times_B B'$ such that $f' \circ \tilde{x} = x$.

Lemma 3.4. *With the notations of Definition 3.3, the morphism* (3.3) *is an isomorphism* $\ker(u) \simeq \ker(u')$. *Moreover, if* $\mathrm{coker}(u) \simeq 0$, *then* $\mathrm{coker}(u') \simeq 0$.

By symmetry of the definition, the same holds with f, f' instead of u, u'.

Proof. Since $f \circ u' = f' \circ u$, we have a natural morphism $\ker u' \to \ker u$ by Lemma 2.27. we can see that this is an inverse to (3.3).

Let us check the second assertion. We pick $g\colon B' \to C$ such that $g \circ u' = 0$. By construction we have an exact sequence $0 \to A \times_B B' \to A \times B' \xrightarrow{s} B \to 0$ and the map u' is induced by $A \times B' \xrightarrow{q} B'$. Since $g \circ u' = 0$, we have a factorization of $g \circ q$ as follows

$$\begin{array}{ccc} A \times B' & \xrightarrow{q} & B' \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle g} \\ B & \xrightarrow{h} & C. \end{array}$$

Let $i\colon A \to A \times B'$ be the natural map. Then $s \circ i = u$ is an epimorphism. But $h \circ s \circ i = g \circ q \circ i = g \circ 0 = 0$. Hence $h = 0$. Hence $g \circ q = 0$, and, since q is an epimorphism (check!), $g = 0$. ☐

Lemma 3.5. *Let* $0 \to A \xrightarrow{i} B \xrightarrow{j} C$ *be an exact sequence in an abelian category and let* $f\colon A' \to A$ *be a morphism. Then we have an exact sequence* $0 \to \operatorname{coker}(f) \xrightarrow{i'} \operatorname{coker}(i \circ f) \xrightarrow{j'} C$.

Proof. The maps i' and j' are uniquely induced by i and j, by definition of a cokernel. Let us check that the kernel of j' is indeed $\operatorname{coker}(f)$.

Let us first assume that the category is **Ab**. We pick $x \in \operatorname{coker}(i \circ f)$ and lift it to $\tilde{x} \in B$. Then $j(\tilde{x}) = j'(x) = 0$. Hence $\tilde{x} \in A$ and $x \in \operatorname{coker}(f)$.

The general case is similar, using Lemma 3.4 (follow the argument on Fig. 1). The element x is replaced by a morphism: We pick $x\colon D \to \operatorname{coker}(i \circ f)$ such that $j' \circ x = 0$. We cannot lift x to $\tilde{x}\colon D \to B$ in general. Instead we set $\widetilde{D} = B \times_{\operatorname{coker}(i \circ f)} D$ and consider the diagram

$$\begin{array}{ccccc} \widetilde{D} & \xrightarrow{\tilde{x}} & B & \xrightarrow{j} & C \\ \downarrow{\scriptstyle p} & & \downarrow & & \| \\ D & \xrightarrow{x} & \operatorname{coker}(i \circ f) & \xrightarrow{j'} & C \end{array}$$

Then $j \circ \tilde{x} = j' \circ x \circ p = 0$ and $\tilde{x}$ factorizes through $\tilde{y}\colon \widetilde{D} \to A$. The morphism $i \circ f\colon A' \to B$ factorizes through $g\colon A' \to \widetilde{D}$ by the remark after (3.3) and Lemma 3.4 gives an exact sequence $A' \xrightarrow{g} \widetilde{D} \xrightarrow{p} D \to 0$. Since $\tilde{x} \circ g = i \circ f$ and i is a monomorphism, we have $\tilde{y} \circ g = f$. Hence $(q \circ \tilde{y}) \circ g = 0$, with $q\colon A \to \operatorname{coker}(f)$ the natural map. Hence $q \circ \tilde{y}$ factorizes through p and $y\colon D \to \operatorname{coker}(f)$. We obtain $i' \circ y = x$ (use that p is an epimorphism).

The uniqueness of y is proved along the same steps. We assume that there exists $z\colon D \to \operatorname{coker}(f)$ such that $i' \circ z = 0$ and we want to prove that $z = 0$. We set $\widehat{D} = A \times_{\operatorname{coker}(f)} D$ and consider the diagram

$$\begin{array}{ccccc} \widehat{D} & \xrightarrow{\tilde{z}} & A & \xrightarrow{\quad i \quad} & B \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q'} \\ D & \xrightarrow{z} & \operatorname{coker}(f) & \xrightarrow{i'} & \operatorname{coker}(i \circ f) \end{array}$$

(Exercise: complete this diagram into a diagram like Fig. 1 to follow the end of the proof.) Since $q' \circ (i \circ \tilde{z}) = 0$ the morphism $i \circ \tilde{z}$ factorizes through $\ker(q')$. Now $\ker(q') = \operatorname{im}(i \circ f)$ (since we work in an abelian category). Since i is a monomorphism we can also check that the morphism $\operatorname{im}(f) \to \operatorname{im}(i \circ f)$ is an isomorphism (exercise!). We deduce that $\tilde{z}$ factorizes though $\operatorname{im}(f) = \ker(q)$. Hence $q \circ \tilde{z} = 0 = z \circ p$. Since p is an epimorphism, we have $z = 0$, as required. □

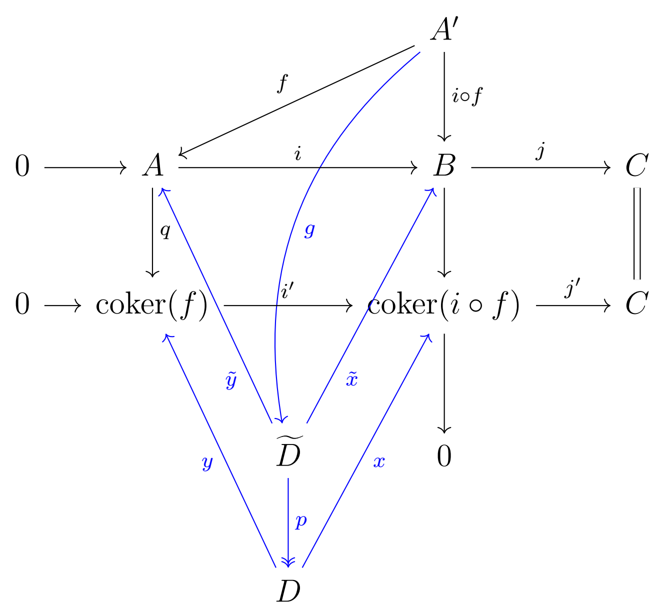


Figure 1. Diagram for the proof Lemma 3.5

Lemma 3.6. *Let $\mathcal{C}$ be an abelian category and let $X = (X^{\cdot}, d_X^{\cdot}) \in \mathbf{C}(\mathcal{C})$. Then we have the exact sequence*

$$0 \to H^i(X) \to \operatorname{coker}(d_X^{i-1}) \to Z^{i+1}(X) \to H^{i+1}(X) \to 0.$$

Proof. By definition of $H^{i+1}(X)$ we have the exact sequence $X^i \xrightarrow{d_X^i} Z^{i+1}(X) \to H^{i+1}(X) \to 0$. Since $d_X^i \circ d_X^{i-1} = 0$, d_X^i factorizes through $\operatorname{coker}(d_X^{i-1})$ and gives the end of the sequence.

We have the exact sequence $0 \to Z^i(X) \to X^i \to Z^{i+1}(X)$ and the map $d_X^{i-1}\colon X^{i-1} \to X^i$ induces $c\colon X^{i-1} \to Z^i(X)$. By Lemma 3.5 we deduce the exact sequence $0 \to H^i(X) \to \operatorname{coker}(d_X^{i-1}) \to Z^{i+1}(X)$, which concludes the proof. □

Here are two useful lemmas to deal with complexes and long cohomology sequences.

Lemma 3.7. *Let $\mathcal{C}$ be an abelian category. We consider the commutative diagram in $\mathcal{C}$*

$$\begin{CD}
0 @>>> X @>f>> Y @>g>> Z \\
@. @VuVV @VvVV @VwVV \\
0 @>>> X' @>f'>> Y' @>g'>> Z'
\end{CD}$$

and we assume that the rows are exact. Then this diagram induces a canonical (in the sense detailed in Proposition 3.9) exact sequence $0 \to \ker u \to \ker v \to \ker w$.

Lemma 3.8 (The snake lemma – see [3] lem. 12.1.1)**.** *Let $\mathcal{C}$ be an abelian category. We consider the commutative diagram in $\mathcal{C}$*

$$\begin{CD}
@. X @>f>> Y @>g>> Z @>>> 0 \\
@. @VuVV @VvVV @VwVV \\
0 @>>> X' @>f'>> Y' @>g'>> Z'
\end{CD}$$

and we assume that the rows are exact. Then this diagram induces a canonical (in the sense detailed in Proposition 3.9) exact sequence

$$\ker u \to \ker v \to \ker w \to \operatorname{coker} u \to \operatorname{coker} v \to \operatorname{coker} w.$$

Proposition 3.9. *Let $\mathcal{C}$ be an abelian category and let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence in $\mathbf{C}(\mathcal{C})$. Then there exists a* canonical *long exact sequence in $\mathcal{C}$*

$$\cdots \to H^n(X) \xrightarrow{H^n(f)} H^n(Y) \xrightarrow{H^n(g)} H^n(Z) \xrightarrow{\delta^n} H^{n+1}(X)$$
$$\xrightarrow{H^{n+1}(f)} H^{n+1}(Y) \xrightarrow{H^{n+1}(g)} H^{n+1}(Z) \to \cdots .$$

Canonical means here: if we have a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \\ 0 & \longrightarrow & X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \longrightarrow & 0, \end{array}$$

then all squares
$$\begin{array}{ccc} H^n(Z) & \xrightarrow{\delta^n} & H^{n+1}(X) \\ {\scriptstyle H^n(w)}\downarrow & & \downarrow{\scriptstyle H^{n+1}(u)} \\ H^n(Z') & \xrightarrow{\delta'^n} & H^{n+1}(X') \end{array}$$
commute.

Proof. For a given i, Lemma 3.7, applied with rows $0 \to X^i \to Y^i \to Z^i$ and $0 \to X^{i+1} \to Y^{i+1} \to Z^{i+1}$ and vertical morphisms $d^i_\bullet$, gives the exact sequence "$E(i)$": $0 \to Z^i(X) \to Z^i(Y) \to Z^i(Z)$. A dual version of Lemma 3.7 would give the exact sequence "$F(i)$": $\operatorname{coker}(d^i_X) \to \operatorname{coker}(d^i_Y) \to \operatorname{coker}(d^i_Z) \to 0$.

We have morphisms $\operatorname{coker} d^{i-1}_\bullet \to Z^{i+1}_\bullet$ (see Lemma 3.6). Lemma 3.8, applied with the rows $F(i-1)$ and $E(i+1)$, together with Lemma 3.6, give the exact sequence of the proposition. □

Lemma 3.10 (The five lemma)**.** *Let $\mathcal{C}$ be an abelian category. We consider the commutative diagram in $\mathcal{C}$*

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle e} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E'. \end{array}$$

We assume that the rows are exact and that a, b, d, e are isomorphisms. Then c is also an isomorphism.

When $\mathcal{C}$ is the category of modules over a ring, the proofs of these results are relatively easy: we can pick an element in an object and follow its images or inverse images by the morphisms in the diagrams ("diagram chasing"). For a general abelian category we can use the Freyd-Mitchell embedding theorem (see [8] section 1.6) which says that any abelian category is a full subcategory of a category of modules over some ring. We can also give direct proofs using Lemma 3.4 and the following one.

Lemma 3.11 (see [3] lem. 8.3.12)**.** *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms in an abelian category such that $g \circ f = 0$. Then this sequence is exact at Y if and only if, for any morphism $h\colon S \to Y$ such that $g \circ f = 0$,* *there exists a commutative diagram*

$$\begin{array}{ccccc} S' & \xrightarrow{f'} & S & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle h} & & \downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z, \end{array}$$

where the first row is exact.

3.1. **Exercises.**

Exercise 3.12. Let $f\colon A \to B$ and $g\colon B \to C$ be morphisms in an abelian category. Prove that there is a unique morphism $h\colon \operatorname{im} f \to \operatorname{im}(g \circ f)$ making the diagram

$$\begin{array}{ccccc} A & \longrightarrow & \operatorname{im} f & \longrightarrow & B \\ & \searrow & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\ & & \operatorname{im}(g \circ f) & \longrightarrow & C \end{array}$$

commute. Prove that h is an epimorphism. We assume now that g is a monomorphism; prove that h is an isomorphism.

4. Functors

Definition 4.1. Let $\mathcal{C}, \mathcal{C}'$ be two categories. A functor F from $\mathcal{C}$ to $\mathcal{C}'$ is the data of maps (also denoted by F) $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}')$ and $F\colon \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}'}(F(X),F(Y))$, for all $X, Y \in \mathrm{Ob}(\mathcal{C})$, satisfying

(i) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all $X \in \mathrm{Ob}(\mathcal{C})$,
(ii) $F(f \circ g) = F(f) \circ F(g)$, for all composable morphisms f, g.

For two functors $F\colon \mathcal{C} \to \mathcal{C}'$ and $G\colon \mathcal{C}' \to \mathcal{C}''$ we define the composition $G \circ F$ by $(G\circ F)(X) = G(F(X))$ for $X \in \mathrm{Ob}(\mathcal{C})$ and $(G \circ F)(f) = G(F(f))$ for all morphisms f in $\mathcal{C}$.

For a category $\mathcal{C}$ we define the *opposite* category $\mathcal{C}^{\mathrm{op}}$ by $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,Y) := \mathrm{Hom}_{\mathcal{C}}(Y,X)$ for all $X, Y \in \mathrm{Ob}(\mathcal{C})$.

A *contravariant* functor from $\mathcal{C}$ to $\mathcal{C}'$ is a functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{C}'$. (Functors can be called covariant functors if we want to insist.)

Example 4.2. Let X be a topological space and let $\mathrm{Op}(X)$ be the category with objects the open subsets of X and morphisms the inclusions, that is, $\mathrm{Hom}_{\mathrm{Op}(X)}(U,V)$ is a set with one object if $U \subset V$ and is empty if $U \not\subset V$. There is only one possibility for the composition law. Then a presheaf on X is a contravariant functor from $\mathrm{Op}(X)$ to **Ab**.

Definition 4.3. Let $\mathcal{C}, \mathcal{C}'$ be two categories and let F, G be two functors from $\mathcal{C}$ to $\mathcal{C}'$. A morphism of functors θ from F to G is the data of morphisms $\theta_X\colon F(X) \to G(X)$ for all $X \in \mathrm{Ob}(\mathcal{C})$ such that, for all morphisms $f\colon X \to Y$ in $\mathcal{C}$, the following diagram commutes

$$\begin{array}{ccc} F(X) & \xrightarrow{\theta_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[\theta_Y]{} & G(Y). \end{array}$$

A functor $F\colon \mathcal{C} \to \mathcal{C}'$ between additive categories is *additive* if the maps $\mathrm{Hom}(A,B) \to \mathrm{Hom}(F(A),F(B))$, $f \mapsto F(f)$, are group morphisms, for all $A, B \in \mathrm{Mod}(\mathbf{k})$.

An additive functor $F\colon \mathcal{C} \to \mathcal{C}'$ between abelian categories is *exact* if it sends short exact sequences to short exact sequences. It is *left exact* if, for any exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C$, the sequence $0 \to F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$ is exact. It is *right exact* if, for any exact sequence $A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the sequence $F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \to 0$ is exact.

Example 4.4. Let X be a topological space and let $U \subset X$ be open. For $F \in \mathbf{Sh}(X)$ we denote by $\Gamma(U; F) = F(U)$ the set of sections over U. Then $\Gamma(U; \cdot)\colon \mathbf{Sh}(X) \to \mathbf{Ab}$ is a left exact functor.

Example 4.5. Let $\mathcal{C}$ be an abelian category and let $M \in \mathrm{Ob}(\mathcal{C})$. Both functors

$$\begin{aligned} \mathrm{Hom}(M, \cdot)\colon \mathcal{C} &\to \mathcal{C} \\ X &\mapsto \mathrm{Hom}(M, X) \end{aligned} \qquad \begin{aligned} \mathrm{Hom}(\cdot, M)\colon \mathcal{C}^{\mathrm{op}} &\to \mathcal{C} \\ X &\mapsto \mathrm{Hom}(X, M) \end{aligned}$$

are left exact.

Definition 4.6. Let $\mathcal{C}$ be an abelian category and let $P \in \mathrm{Ob}(\mathcal{C})$. We say that P is *projective* if the functor $\mathrm{Hom}(P, \cdot)$ is exact, that is, if for any short exact sequence $A \to B \to 0$, the sequence $\mathrm{Hom}(P, A) \to \mathrm{Hom}(P, B) \to 0$ is exact. We say that $\mathcal{C}$ has *enough projectives* if for any $M \in \mathrm{Ob}(\mathcal{C})$, there exist a projective object P and an exact sequence $P \to M \to 0$.

Let $I \in \mathrm{Ob}(\mathcal{C})$. We say that I is *injective* if the functor $\mathrm{Hom}(\cdot, I)$ is exact, that is, if for any short exact sequence $0 \to A \to B$, the sequence $\mathrm{Hom}(A, I) \to \mathrm{Hom}(B, I) \to 0$ is exact. We say that $\mathcal{C}$ has *enough injectives* if for any $M \in \mathrm{Ob}(\mathcal{C})$, there exist an injective object I and an exact sequence $0 \to M \to I$.

We will see that the category $\mathrm{Mod}(\mathbf{k})$ for a ring $\mathbf{k}$ has enough projectives and enough injectives. However the category $\mathbf{Sh}(X)$ has no non-zero projectives in general, but it has enough injectives.

Let $\mathcal{C}$ be an abelian category and let $M \in \mathrm{Ob}(\mathcal{C})$. A *left resolution* of M is a complex $X = (X^{\cdot}, d_X^{\cdot}) \in \mathbf{C}(\mathcal{C})$ such that $X^i \simeq 0$ for $i > 0$, together with a morphism $\varepsilon\colon X^0 \to M$ such that the sequence

$$\cdots \to X^{-2} \xrightarrow{d^{-2}} X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{\varepsilon} M \to 0$$

is exact. In particular $H^i(X) \simeq 0$ for all $i \neq 0$ and $H^0(X) \simeq M$. A resolution is called projective if all the X^i's are projective.

A *right resolution* is defined by reversing the arrows (hence we have an exact sequence $0 \to M \xrightarrow{\varepsilon} X^0 \xrightarrow{d^0} \cdots$). It is called injective if all the X^i's are injective.

Proposition 4.7. *Let $\mathcal{C}$ be an abelian category. We assume that $\mathcal{C}$ has enough projectives. Then any $M \in \mathrm{Ob}(\mathcal{C})$ has a projective (left) resolution.*

Theorem 4.8. *Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a right exact functor between abelian categories. Let $M \in \mathcal{C}$ be an object which has a projective resolution*

$\cdots \to P^{-2} \xrightarrow{d^{-2}} P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{\varepsilon} M \to 0$. *Then, for* $i \in \mathbb{N}$,

$$L^iF(M) := H^{-i}\Big(\cdots \to F(P^{-2}) \xrightarrow{F(d^{-2})} F(P^{-1}) \xrightarrow{F(d^{-1})} F(P^0) \to 0\Big)$$

is independent of the choice of the projective resolution.

If $\mathcal{C}$ *has enough projectives, this defines a sequence of functors* $L^iF : \mathcal{C} \to \mathcal{C}'$, *with the property: for any short exact sequence* $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ *in* $\mathcal{C}$, *we have a long exact sequence in* $\mathcal{C}'$

$$L^iF(M') \xrightarrow{L^iF(u)} L^iF(M) \xrightarrow{L^iF(v)} L^iF(M'') \xrightarrow{\delta^i} L^{i-1}F(M') \\ \to \cdots \to L^1F(M'') \to F(M') \to F(M) \to F(M'') \to 0.$$

5. Exercises

Exercise 5.1. Let $f\colon A \to B$, $g\colon B \to C$ be morphisms in an abelian category. Prove that there exists a unique morphism $f'\colon \operatorname{im}(g \circ f) \to \operatorname{im}(g)$ such that we have the commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & \operatorname{im}(g\circ f) & \longrightarrow & C \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & \nearrow & \\ B & \longrightarrow & \operatorname{im}(g) & & \end{array}$$

Prove that f' is a monomorphism. Assume now that f is an epimorphism; prove that f' is an isomorphism.

Exercise 5.2. We prove Lemma 3.11: Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms in an abelian category such that $g \circ f = 0$. Then this sequence is exact at Y if and only if, for any morphism $h\colon S \to Y$ such that $g \circ f = 0$, there exists a commutative diagram

$$\begin{array}{ccccc} S' & \xrightarrow{f'} & S & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle h} & & \downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z, \end{array}$$

where the first row is exact.

(i) Assume the sequence is exact and pick h as above. Then h factorizes through $\ker(g)$. Prove that $S' = X \times_{\ker(g)} S$ satisfies the condition.

(ii) We prove the opposite direction. Assume first the category is **Ab**; what do you choose for S to prove the result? In general choose the "same" object for S and use the above diagram to build an inverse to the natural morphism $\operatorname{im}(f) \to \ker(g)$. Hint:

$$\begin{array}{ccccccc} \ker(f') & \longrightarrow & S' & \xrightarrow{f'} & S & \longrightarrow & 0 \\ & & \downarrow & \swarrow & \downarrow{\scriptstyle h} & & \downarrow \\ & & \operatorname{im}(f) & \longrightarrow & Y & \xrightarrow{g} & Z, \end{array}$$

Exercise 5.3. In **Ab** prove that $\mathbb{Z}$ is projective. More generally, for any family I, the free abelian group $\mathbb{Z}^{(I)}$ is projective. Deduce that **Ab** has enough projectives.

Exercise 5.4. Let $\mathbf{k}$ be a field and **Vect** the category of $\mathbf{k}$-vector spaces. Prove that any object in **Vect** is projective and injective.

Exercise 5.5. We use the notations of Exercise 2.36: $\mathcal{C}$ is an abelian category and we consider the category $\operatorname{Mor}(\mathcal{C})$ of morphisms in $\mathcal{C}$. We have seen that it is abelian. Prove that $\ker\colon \mathrm{Mor}(\mathcal{C}) \to \mathcal{C}$, $(X \xrightarrow{u} X') \mapsto \ker(u)$ is a left exact functor. Give an example (with $\mathcal{C} = \mathbf{Ab}$) to see that it is not right exact in general.

Exercise 5.6. When $\mathcal{C} = \mathbf{Vect}$ (we fix a field), check that the objects $(X \xrightarrow{\mathrm{id}} X)$ and $(X \xrightarrow{0} 0)$ are injective in $\mathrm{Mor}(\mathcal{C})$. Similarly, the objects $(X \xrightarrow{\mathrm{id}} X)$ and $(0 \xrightarrow{0} X)$ are projective.

As in the previous exercise $\mathrm{coker}\colon \mathrm{Mor}(\mathcal{C}) \to \mathcal{C}$ is a right exact functor. Compute $L^i \,\mathrm{coker}$.

Exercise 5.7. Let X be a topological space and $F \in \mathbf{Sh}(X)$. Let F_x be the germ at x and let F^x be the corresponding skyscrapper sheaf, that is, $F^x = (F_x)_{\{x\}}$. We have a natural morphism $i^x\colon F \to F^x$ such that $(i^x)_x = \mathrm{id}_{F_x}$.

We set $\widetilde{F} = \prod_{x\in X} F^x$. Describe the section $\widetilde{F}(U)$ for an open set U.

Prove that the morphism $i\colon F \to \widetilde{F}$ induced by the i^x is a monomorphism.

Exercise 5.8. Let $\mathcal{C}$ be an abelian category. Let I_a, $a \in A$, be injective objects in $\mathcal{C}$. We assume that $\prod_{a\in A} I_a$ exists in $\mathcal{C}$. Prove that $\prod_{a\in A} I_a$ is injective.

Exercise 5.9. We choose a field $\mathbf{k}$. We defines $\mathbf{Sh}_{\mathbf{k}}(X)$ exactly like $\mathbf{Sh}(X)$ replacing everywhere "abelian group" by "$\mathbf{k}$-vector space" and "additive map" by "linear map". Let A be a $\mathbf{k}$-vector space and $x \in X$. Prove that the skyscrapper sheaf $A_{\{x\}}$ is injective. Deduce that the object $\widetilde{F}$ above is injective.

Exercise 5.10. Let X be a topological space and $Z \subset X$ a closed subset. Let A be a group. We already defined A_Z by $A_Z(U) = \{f\colon Z\cap U \to A,\ f \text{ is locally constant}\}$. Let $W \subset X$ be **locally closed**, which means that W is difference of two closed subsets $W = Z \setminus Y$ for $Y \subset Z$. We have a natural morphism $A_Z \to A_Y$. We set $A_W = \ker(A_Z \to A_Y)$.

Check that A_W is independent of Y, Z. Hint: we can reduce to the case $Z = \overline{W}$ and $Y = \overline{W} \setminus W$.

Prove that $(A_W)_x = A$ if $x \in W$ and $(A_W)_x = 0$ if $x \notin W$.

For $X = \mathbb{R}$ and $W = [0,1[$, describe $A_W(]a,b[)$ according to the positions of a, b with respect to $0, 1$.

Exercise 5.11. Let $W \subset W' \subset X$ be locally closed subsets. Prove that

if W is closed in W', there is a natural morphism $r\colon A_{W'} \to A_W$ such that r_x is an isomorphism when $x \in W$,

if W is open in W', there is a natural morphism $i\colon A_W \to A_{W'}$ such that i_x is an isomorphism when $x \in W$.

Prove that we obtain an exact sequence $0 \to A_{W'\setminus W} \to A_{W'} \to A_W \to 0$.

Exercise 5.12. We consider sheaves on $\mathbb{R}$. Let A be an abelian group. Prove that $\mathrm{Hom}(A_{[0,1]}, A_{\mathbb{R}}) = 0$ and $\mathrm{Hom}(A_{\mathbb{R}}, A_{]0,1[}) = 0$.

Exercise 5.13. Let X be a topological space and let $B = \{B_i\}_{i\in I}$ a basis of open subsets. Let $\mathbf{Sh}_B(X)$ be the category of "sheaves over B" where a sheaf F over B is the same data as a sheaf but we only consider the $F(B_i)$ and the restriction maps $F(B_i) \to F(B_j)$, $i, j \in I$; the separation condition is the same and the gluing condition becomes: if $B_{i_0} = \bigcup_{k\in K} B_k$ and $s_k \in F(B_k)$ satisfy $s_k|_{B_k} = s_l|_{B_l}$ for all $k, l \in K$ and $i \in I$, then there exists $s \in F(B_{i_0})$ such that $s|_{B_k} = s_k$. The morphisms are also defined only for the B_i's.

We have an obvious forget functor $\mathrm{For}\colon \mathbf{Sh}(X) \to \mathbf{Sh}_B(X)$. We want to prove that it is an equivalence.

We construct an inverse functor. Let $F \in \mathbf{Sh}_B(X)$. We want $G \in \mathbf{Sh}(X)$ such that $G(B_i) = F(B_i)$ for all $i \in I$. Let $U \subset X$ be an open subset and write $U = \bigcup_{k\in K} B_k$ for some $K \subset I$. We set

$$G_K(U) = \{(s_k)_{k\in K};\ s_k \in F(B_k),\ s_k|_{B_i} = s_l|_{B_i} \text{ for all } k, l \in K,\ i \in I\}.$$

We must have $G(U) = G_K(U)$, but we need to check that $G_K(U)$ is independent of the covering K. We first assume that we have another covering L **refining** K, which means that, for each $l \in L$ there exists $k \in K$ such that $B_l \subset B_k$.

Let $s = (s_k)_{k\in K}$ in $G_K(U)$ be given and $i \in I$ such that $B_i \subset B_k$ for some $k \in K$. Then $s_k|_{B_i}$ is independent of k (that is, if $B_i \subset B_l$ for another $l \in K$ then $s_k|_{B_i} = s_l|_{B_i}$). Hence we may define $s|_{B_i} := s_k|_{B_i}$. Using these notations we define $r^L_K\colon G_K(U) \to G_L(U)$ by $s \mapsto (s_l)_{l\in L}$, where $s_l = s|_{B_l}$. Check that r^L_K is well-defined and is an isomorphism.

Check that, for two coverings $U = \bigcup_{k\in K} B_k$ and $U = \bigcup_{k\in K'} B_k$ there exists a third covering L which refines both K and K'. We then have canonical isomorphisms

$$G_K(U) \xrightarrow{\sim} G_L(U) \xleftarrow{\sim} G_{K'}(U)$$

which proves that $G_K(U)$ does not depend on K. We set $G(U) = G_K(U)$, for any covering K. We have in particular $G(B_i) = F(B_i)$ for all $i \in I$.

Check that G is a sheaf.

The construction $F \to G$ then gives a functor $\mathbf{Sh}_B(X) \to \mathbf{Sh}(X)$. Check that it is an inverse to $\mathbf{Sh}(X) \to \mathbf{Sh}_B(X)$.

6. More on functors and resolutions

Definition 6.1. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a functor. We say that F is full (resp. faithful, fully faithful) if the maps $F\colon \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}'}(F(X),F(Y))$ are surjective (resp. injective, bijective), for all $X,Y \in \mathrm{Ob}(\mathcal{C})$.

We say that F is essentially surjective if for each $Y \in \mathrm{Ob}(\mathcal{C}')$ there exist $X \in \mathrm{Ob}(\mathcal{C})$ and an isomorphism $F(X) \simeq Y$.

We say that F is an equivalence of categories if there exist a functor $G\colon \mathcal{C}' \to \mathcal{C}$ and isomorphisms of functors $\mathrm{id}_{\mathcal{C}} \simeq G \circ F$ and $\mathrm{id}_{\mathcal{C}'} \simeq F \circ G$. We then write $F\colon \mathcal{C} \xrightarrow{\sim} \mathcal{C}'$ and we say that F and G are quasi-inverse to each other.

For example the category of finite dimensional vector spaces over some field $\mathbf{k}$, say $\mathbf{Vect}_f(\mathbf{k})$, is equivalent to its *full* subcategory $\mathbf{Mat}(\mathbf{k})$ with $\mathrm{Ob}(\mathbf{Mat}(\mathbf{k})) = \{\mathbf{k}^n;\, n \in \mathbb{N}\}$ (where full means that the Hom sets are the same: $\mathrm{Hom}_{\mathbf{Mat}(\mathbf{k})}(\mathbf{k}^n,\mathbf{k}^m) = \mathrm{Hom}_{\mathbf{Vect}_f(\mathbf{k})}(\mathbf{k}^n,\mathbf{k}^m) = \mathrm{Mat}(m \times n,\mathbf{k})$).

Proposition 6.2. *A functor $F\colon \mathcal{C} \to \mathcal{C}'$ is an equivalence of categories if and only if it is fully faithful and essentially surjective.*

Let $\mathcal{C}$ be an abelian category and let $M \in \mathrm{Ob}(\mathcal{C})$. A *left resolution* of M is a complex $X = (X^{\cdot}, d_X^{\cdot}) \in \mathbf{C}(\mathcal{C})$ such that $X^i \simeq 0$ for $i > 0$, together with a morphism $\varepsilon\colon X^0 \to M$ such that the sequence

$$\cdots \to X^{-2} \xrightarrow{d^{-2}} X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{\varepsilon} M \to 0$$

is exact. In particular $H^i(X) \simeq 0$ for all $i \neq 0$ and $H^0(X) \simeq M$. A resolution is called projective if all the X^i's are projective.

A *right resolution* is defined by reversing the arrows (hence we have an exact sequence $0 \to M \xrightarrow{\varepsilon} X^0 \xrightarrow{d^0} \cdots$). It is called injective if all the X^i's are injective.

Proposition 6.3. *Let $\mathcal{C}$ be an abelian category. We assume that $\mathcal{C}$ has enough projectives. Then any $M \in \mathrm{Ob}(\mathcal{C})$ has a projective (left) resolution. Similarly, when we have enough injectives, we have injective right resolutions.*

Proof. By definition of "enough projectives" there exists an epimorphism $P^0 \xrightarrow{\varepsilon} M$ with P^0 projective. We set $M^1 = \ker \varepsilon \xrightarrow{i^1} P^0$ and choose an epimorphism $P^{-1} \xrightarrow{e^1} M^1$. We set $d^{-1} = i^1 \circ e^1$. We then have the exact sequence $P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{\varepsilon} M \to 0$. We set $M^2 = \ker d^{-1}$ and proceed with M^2 as with M^1. We go on and obtain the resolution by induction. □

The next proposition says that a projective resolution is unique up to homotopy in the following sense.

Definition 6.4. Let $\mathcal{C}$ be an additive category and let $P = (P^{\cdot}, d_P^{\cdot})$, $Q = (Q^{\cdot}, d_Q^{\cdot}) \in \mathbf{C}(\mathcal{C})$. We say that two morphisms $f, g\colon P \to Q$ in $\mathbf{C}(\mathcal{C})$ are homotopic if there exists a family of morphisms $s^i\colon P^i \to Q^{i-1}$, $i \in \mathbb{Z}$, such that

$$f^n - g^n = d_Q^{n-1} \circ s^n + s^{n+1} \circ d_P^n,$$

for all $n \in \mathbb{Z}$.

The homotopy relation is compatible with the additive structure of $\mathrm{Hom}(P, Q)$ and with the composition in $\mathbf{C}(\mathcal{C})$. It follows that we can define a category of *complexes up to homotopy* as follows.

Definition 6.5. Let $\mathcal{C}$ be an additive category. We define a category $\mathbf{K}(\mathcal{C})$ by $\mathrm{Ob}(\mathbf{K}(\mathcal{C})) = \mathrm{Ob}(\mathbf{C}(\mathcal{C}))$ and

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{C})}(P, Q) = \mathrm{Hom}_{\mathbf{C}(\mathcal{C})}(P, Q)/\sim_h,$$

where $\sim_h$ is the homotopy relation on $\mathrm{Hom}_{\mathbf{C}(\mathcal{C})}(P, Q)$. Then $\mathbf{K}(\mathcal{C})$ is an additive category.

We have an obvious functor $\mathrm{Ob}(\mathbf{K}(\mathcal{C})) \to \mathrm{Ob}(\mathbf{C}(\mathcal{C}))$ which is the identity on objects and the quotient map on the morphisms.

Proposition 6.6. *Let $\mathcal{C}$ be an abelian category, let $M \in \mathrm{Ob}(\mathcal{C})$ and let $P = (P^{\cdot}, d_P^{\cdot}) \in \mathbf{C}(\mathcal{C})$ together with $\varepsilon\colon P^0 \to M$ be a projective resolution of M. Let $f'\colon M \to N$ be a morphism in $\mathcal{C}$. Let $X = (X^{\cdot}, d_X^{\cdot}) \in \mathbf{C}(\mathcal{C})$ together with $\eta\colon X^0 \to N$ be a left resolution of N. Then there exists a morphism $f\colon P \to X$ in $\mathbf{C}(\mathcal{C})$ lifting f' in the sense that $f' \circ \varepsilon = \eta \circ f^0$. In other words there exists a commutative diagram*

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & P^{-2} & \xrightarrow{d_P^{-2}} & P^{-1} & \xrightarrow{d_P^{-1}} & P^0 & \xrightarrow{\varepsilon} & M \\
 & & \downarrow{\scriptstyle f^{-2}} & & \downarrow{\scriptstyle f^{-1}} & & \downarrow{\scriptstyle f^{0}} & & \downarrow{\scriptstyle f'} \\
\cdots & \longrightarrow & X^{-2} & \xrightarrow[d_X^{-2}]{} & X^{-1} & \xrightarrow[d_X^{-1}]{} & X^0 & \xrightarrow[\eta]{} & N.
\end{array}$$

Moreover, if $g\colon P \to X$ is another morphism lifting f', then f and g are homotopic.

Proof. (i) The existence of f^0 follows from the facts that η is an epimorphism and P^0 is projective. Then we remark that $f^0 \circ d_P^{-1}$ factorizes through $\ker \eta = \mathrm{im}\, d_X^{-1}$. Hence we obtain f^{-1} in the same way, using the facts that d_X^{-1} is an epimorphism to its image and P^{-1} is projective. We obtain all f^k in this way inductively.

(ii) We set $h^k = f^k - g^k$. We have the commutative diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & P^{-2} & \xrightarrow{d_P^{-2}} & P^{-1} & \xrightarrow{d_P^{-1}} & P^0 & \xrightarrow{\varepsilon} & M \\
 & \downarrow{\scriptstyle h^{-2}} & & \downarrow{\scriptstyle h^{-1}} & & \downarrow{\scriptstyle h^{0}} & & \downarrow{\scriptstyle 0} \\
\cdots \longrightarrow & X^{-2} & \xrightarrow[d_X^{-2}]{} & X^{-1} & \xrightarrow[d_X^{-1}]{} & X^0 & \xrightarrow[\eta]{} & N.
\end{array}$$

Since $\eta \circ h^0 = 0$, h^0 factorizes through $i^0 \colon P^0 \to \ker \eta = \operatorname{im} d_X^{-1}$. Since $X^{-1} \to \operatorname{im} d_X^{-1}$ and P^0 is projective, we can lift i^0 to $s^0 \colon P^0 \to X^{-1}$. We then have $h^0 = d_X^{-1} \circ s^0$.

We define $h'^{-1} = h^{-1} - s^0 \circ d_P^{-1}$. Then $d_X^{-1} \circ h'^{-1} = 0$ and we can apply the same procedure to find $s^{-1} \colon P^{-1} \to X^{-2}$ such that $h'^{-1} = d_X^{-2} \circ s^{-1}$. Hence $h^{-1} = s^0 \circ d_P^{-1} + d_X^{-2} \circ s^{-1}$. Now we go on inductively. $\square$

We obtain the injective versions of Propositions 6.3 and 6.6 by reversing the arrows.

Definition 6.7. Let $F \colon \mathcal{C} \to \mathcal{C}'$ be an additive functor between additive categories. We define $\mathbf{C}(F) \colon \mathbf{C}(\mathcal{C}) \to \mathbf{C}(\mathcal{C}')$ by $F(X^{\cdot}, d_X^{\cdot}) = (F(X^{\cdot}), F(d_X^{\cdot}))$. Then $\mathbf{C}(F)$ is additive and compatible with homotopy. It induces an additive functor $\mathbf{K}(F) \colon \mathbf{K}(\mathcal{C}) \to \mathbf{K}(\mathcal{C}')$.

It is easy to check that two homotopic morphisms of complexes induce the same morphism on homology (when $\mathcal{C}$ is abelian):

Lemma 6.8. *Let $\mathcal{C}$ be an abelian category and let $f \colon X \to Y$ be a morphism in $\mathbf{C}(\mathcal{C})$. We assume that f is homotopic to the zero morphism. Then $H^i(f) \colon H^i(X) \to H^i(Y)$ is zero, for all $i \in \mathbb{Z}$. In particular the homology functors $H^i \colon \mathbf{C}(\mathcal{C}) \to \mathcal{C}$ induce well-defined functors $H^i \colon \mathbf{K}(\mathcal{C}) \to \mathcal{C}$.*

Definition 6.9. Let $F \colon \mathcal{C} \to \mathcal{C}'$ be a right exact functor between two abelian categories. We assume that $\mathcal{C}$ has enough projectives. For $M \in \mathcal{C}$ and $i \in \mathbb{Z}$ we define $L^iF(M) = H^i(\mathbf{K}(F)(P))$, where P is any projective resolution of M.

The fact that $L^iF(M)$ is well-defined up to a unique isomorphism follows from Proposition 6.6. The right exactness of F ensures that

$$L^0F(M) \simeq F(M), \qquad \text{for all } M \in \operatorname{Ob}(\mathcal{C}).$$

To see that L^iF is a functor we give a slightly different version of Propositions 6.3 and 6.6.

Let $\mathbf{K}_{pr}(\mathcal{C})$ be the full subcategory of $\mathbf{K}(\mathcal{C})$ formed by the complexes $P = (P^{\cdot}, d_P^{\cdot})$ such that $P^i = 0$ for $i > 0$, $H^i(P) \simeq 0$ for all $i \neq 0$ and P^i is projective for each $i \leq 0$ ("*pr*" stands for "projective resolution"). Then Propositions 6.3 and 6.6 give

Proposition 6.10. *The functor $H^0\colon \mathbf{K}_{pr}(\mathcal{C}) \to \mathcal{C}$ is essentially surjective and fully faithful. In other words, it is an equivalence and we can find a quasi-inverse $\mathbf{res}_{pr}\colon \mathcal{C} \to \mathbf{K}_{pr}(\mathcal{C})$.*

Now we choose $\mathbf{res}_{pr}$ as in the proposition and we can rephrase Definition 6.9 by $L^iF(M) = H^i(\mathbf{K}(F)(\mathbf{res}_{pr}(M)))$, which shows that L^iF is a functor.

Different choices of inverse to $H^0\colon \mathbf{K}_{pr}(\mathcal{C}) \to \mathcal{C}$ give different functors $\mathbf{res}_{pr}$ but they are (canonically) isomorphic by the following remark.

Remark 6.11. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between two categories which is an equivalence of categories. Let $G_1, G_2\colon \mathcal{D} \to \mathcal{C}$ be inverses of F, together with isomorphisms of functors $\varepsilon_i\colon F \circ G_i \xrightarrow{\sim} \mathrm{id}_{\mathcal{D}}$. Then there exists a unique isomorphism of functors $\varepsilon\colon G_1 \xrightarrow{\sim} G_2$ such that $F \circ \varepsilon = \varepsilon_2 \circ \varepsilon_1^{-1}$, where $F \circ \varepsilon$ denotes abusively the morphism given by $(F \circ \varepsilon)(X) = F(\varepsilon(X))$, for $X \in \mathcal{C}$.

Indeed, we must define $\varepsilon(Y)\colon G_1(Y) \to G_2(Y)$, $Y \in \mathcal{D}$, as the inverse image of $\varepsilon_2(Y) \circ \varepsilon_1^{-1}(Y)$ by the bijection $\mathrm{Hom}(G_1(Y), G_2(Y)) \xrightarrow{\sim} \mathrm{Hom}(F \circ G_1(Y), F \circ G_2(Y))$, and we can check that this gives an isomorphism of functors.

Proposition 6.12. *Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence in $\mathcal{C}$. Let $P = (P^\cdot, d_P^\cdot) \in \mathbf{C}(\mathcal{C})$ together with $\varepsilon\colon P^0 \to X$ be a projective resolution of X. Let $R = (R^\cdot, d_R^\cdot) \in \mathbf{C}(\mathcal{C})$ together with $\gamma\colon R^0 \to Z$ be a projective resolution of Z. We set $Q^k = P^k \oplus R^k$. Then we can find a differential $d_Q^\cdot$ and $\eta\colon Q^0 \to Y$ turning $(Q^\cdot, d_Q^\cdot)$ into a projective resolution of Y such that the natural morphisms $i_k\colon P^k \to Q^k$, $p_k\colon Q^k \to R^k$ give a commutative diagram of resolutions:*

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & P^{-2} & \xrightarrow{d_P^{-2}} & P^{-1} & \xrightarrow{d_P^{-1}} & P^0 & \xrightarrow{\varepsilon} & X \\
 & & \downarrow{\scriptstyle i^{-2}} & & \downarrow{\scriptstyle i^{-1}} & & \downarrow{\scriptstyle i^{0}} & & \downarrow{\scriptstyle f} \\
\cdots & \dashrightarrow & Q^{-2} & \overset{d_Q^{-2}}{\dashrightarrow} & Q^{-1} & \overset{d_Q^{-1}}{\dashrightarrow} & Q^0 & \overset{\gamma}{\dashrightarrow} & Y \\
 & & \downarrow{\scriptstyle p^{-2}} & & \downarrow{\scriptstyle p^{-1}} & & \downarrow{\scriptstyle p^{0}} & & \downarrow{\scriptstyle g} \\
\cdots & \longrightarrow & R^{-2} & \xrightarrow[d_R^{-2}]{} & R^{-1} & \xrightarrow[d_R^{-1}]{} & R^0 & \xrightarrow[\eta]{} & Z.
\end{array}
$$

Proof. Since g is an epimorphism and R^0 is projective, we can factorize g through $\eta'\colon R^0 \to Y$. Then $\gamma = (f \circ \varepsilon, \eta')\colon Q^0 = P^0 \oplus R^0 \to Y$ gives the commutative squares. Moreover γ is an epimorphism: if $a\colon Y \to M$ is such that $a \circ \gamma = 0$, then $a \circ f \circ \varepsilon = 0$, hence $a \circ f = 0$ (because ε is an epimorphism), hence a factorizes through g by $a'\colon Z \to M$; we then obtain $a' \circ \eta = 0$, hence $a' = 0$ (since η is an epimorphism), hence $a = 0$.

The snake lemma gives the exact sequence $0 \to \ker \varepsilon \to \ker \gamma \to \ker \eta \to 0$. We replace the initial exact sequence by this one and P^0, Q^0, R^0 by P^{-1}, Q^{-1}, R^{-1}. The same argument gives an epimorphism $e^{-1} \colon Q^{-1} \to \ker \gamma$ making commutative squares. We let d_Q^{-1} be the composition of e^{-1} and the morphism $\ker \gamma \to Q^0$.

Now we go on by induction. □

Theorem 6.13. *Let $F \colon \mathcal{C} \to \mathcal{C}'$ be a right exact functor between two abelian categories. We assume that $\mathcal{C}$ has enough projectives. Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence in $\mathcal{C}$. Then there exists a canonical long exact sequence in $\mathcal{C}'$*

$$\cdots \to L^nF(X) \xrightarrow{L^nF(f)} L^nF(Y) \xrightarrow{L^nF(g)} L^nF(Z) \xrightarrow{\delta^n} L^{n+1}F(X) \xrightarrow{L^{n+1}F(f)} L^{n+1}F(Y) \xrightarrow{L^{n+1}F(g)} L^{n+1}F(Z) \to \cdots .$$

More precisely, if we have a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \to & 0 \\ & & \downarrow u & & \downarrow v & & \downarrow w & & \\ 0 & \to & X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \to & 0, \end{array}$$

then all squares

$$\begin{array}{ccc} L^nF(Z) & \xrightarrow{\delta^n} & L^{n+1}F(X) \\ \downarrow L^nF(w) & & \downarrow L^nF(u) \\ L^nF(Z') & \xrightarrow{\delta'^n} & L^{n+1}F(X') \end{array}$$

commute.

Proof. We use the result and the notations of Proposition 6.12. The morphisms $i^{\cdot} \colon P^{\cdot} \to Q^{\cdot}$ and $p^{\cdot} \colon Q^{\cdot} \to R^{\cdot}$ in $C(\mathcal{C})$ give the exact sequence $0 \to P^{\cdot} \to Q^{\cdot} \to R^{\cdot} \to 0$. Indeed this is a general fact (even in an additive category) that the natural morphism $P^i \to P^i \oplus R^i$ is a monomorphism with cokernel R^i. Now we apply the functor $C(F)$ to the sequence and obtain a sequence in $C(\mathcal{C}')$: $0 \to F(P^{\cdot}) \to F(Q^{\cdot}) \to F(R^{\cdot}) \to 0$. Since F is additive, we have $F(Q^i) \simeq F(P^i) \oplus F(R^i)$ and the morphisms still are the natural morphisms from/to a product/sum. Hence this sequence in $C(\mathcal{C}')$ is alos exact. Now the theorem follows from Proposition 3.9. □

Actually we only used the additivity of F in the proof of the theorem. The hypothesis that F is right exact is needed to make the connection between F and L^0F:

Lemma 6.14. *With the hypothesis of Theorem 6.13 we have $L^0F \simeq F$.*

Proof. Let $X \in \mathcal{C}$ be given with a projective resolution $P^\cdot$ together with $\varepsilon\colon P^0 \to X$ such that $\cdots \to P^{-1} \to P^0 \to X \to 0$ is exact. The hypothesis says that $F(P^{-1}) \to F(P^0) \to F(X) \to 0$ is exact, which means $F(X) \simeq \operatorname{coker}(F(d_P^{-1}))$. On the other hand $H^0(\cdots \to F(P^{-1}) \to F(P^0) \to 0) \simeq \operatorname{coker}(F(d_P^{-1}))$ by definition, which gives the result. □

Definition 6.9, Proposition 6.10 and Theorem 6.13 have analogs for left exact functors in the case where $\mathcal{C}$ has enough injectives. In particular we can define $\mathbf{K}_{ir}(\mathcal{C})$ to be the full subcategory of $\mathbf{K}(\mathcal{C})$ formed by the complexes $I = (I^\cdot, d_I^\cdot)$ such that $I^i = 0$ for $i < 0$, $H^i(I) \simeq 0$ for all $i \neq 0$ and I^i is injective for each $i \geq 0$ ("ir" stands for "injective resolution"). Then $H^0\colon \mathbf{K}_{ir}(\mathcal{C}) \to \mathcal{C}$ is essentially surjective and fully faithful as in Proposition 6.10 and we can find a quasi-inverse $\mathbf{res}_{ir}\colon \mathcal{C} \to \mathbf{K}_{ir}(\mathcal{C})$. If $F\colon \mathcal{C} \to \mathcal{C}'$ is a left exact functor, we define $R^iF(M) = H^i(\mathbf{K}(F)(\mathbf{res}_{ir}(M)))$. Since F is left exact, we can see that $R^0F = F$. We then have an analog of Theorem 6.13 by replacing all L^n by R^n.

Example 6.15. Let G be a group and $F = (-)_G\colon G - \mathrm{Mod} \to \mathbf{Ab}$ the functor of coinvariants. We have seen that it is right exact. Let us compute $L^iF(\mathbb{Z})$, where $\mathbb{Z}$ is the trivial representation, when $G = \mathbb{Z}/n\mathbb{Z}$ is a finite cyclic group.

We first define the group ring of G, for any group G. Let $\mathbb{Z}[G]$ be the free abelian group generated by the set G, which means $\mathbb{Z}[G] = \mathbb{Z}^{(G)}$, or, $\mathbb{Z}[G] = \{\sum_{g\in G} n_g e_g\}$, where $\{e_g\}$ is the canonical base (if there is no ambiguity, we may even write g instead of e_g) and the n_g in the sum are all 0 but a finite number of them. Then $\mathbb{Z}[G]$ is an abelian group for the termwise sum and also a ring, where the multiplication is induced by the relation $e_g e_h = e_{gh}$. In other words $(\sum_{g\in G} n_g e_g)(\sum_{g\in G} n'_g e_g) = \sum_{g\in G}(\sum_{h\in G} n_h n'_{h^{-1}g}) e_g$.

Now $\mathbb{Z}[G]$ is also a G-module for the action $g \cdot x = e_g x$ (it is called the *regular* representation of G). We can see that, for any $M \in G - \mathrm{Mod}$, the map

$$\mathrm{Hom}_{G-\mathrm{Mod}}(\mathbb{Z}[G], M) \to M \qquad u \mapsto u(e_{1_G}),$$

is an isomorphism of abelian groups. It follows easily that $\mathbb{Z}[G]$ is projective.

Now we assume $G = \mathbb{Z}/n\mathbb{Z}$ and define $N = \sum_g e_g \in \mathbb{Z}[G]$ (the *norm* element) and $\delta = e_0 - e_1$. We see that $e_g N = N e_g = N$ for any g, hence $N\delta = \delta N = 0$. We even have the exact sequence

$$\cdots \xrightarrow{\cdot\delta} \mathbb{Z}[G] \xrightarrow{\cdot N} \mathbb{Z}[G] \xrightarrow{\cdot\delta} \mathbb{Z}[G] \xrightarrow{\cdot N} \mathbb{Z}[G] \xrightarrow{\cdot\delta} \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

where $\varepsilon(\sum_{g\in G} n_g e_g) = \sum n_g$, and $\cdot N$, $\cdot\delta$ are the multiplication on the right (which are morphisms of left modules).

Now $(\mathbb{Z}[G])_G \simeq \mathbb{Z}$ is the quotient of $\mathbb{Z}[G]$ by $\mathrm{im}(\cdot\delta)$. It follows that

$$H_i(G,\mathbb{Z}) \simeq H^{-i}\Big(\cdots \xrightarrow{\bar\delta} \mathbb{Z} \xrightarrow{\bar N} \mathbb{Z} \xrightarrow{\bar\delta} \mathbb{Z} \xrightarrow{\bar N} \mathbb{Z} \xrightarrow{\bar\delta} \mathbb{Z}),$$

where $\bar\delta$ and $\bar N$ are the maps induced by $\cdot\delta$ and $\cdot N$. We see that $\bar\delta = 0$ and $\bar N$ is the multiplication by n. Hence

$$H_i(G,\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ \mathbb{Z}/n\mathbb{Z} & \text{if } i = 1,3,5,\dots, \\ 0 & \text{if } i = 2,4,6,\dots. \end{cases}$$

Remark 6.16. In Example 6.15 to see that $\mathbb{Z}[G]$ we can use the more general fact that in the category of modules over a ring R, the free module $R^{(I)}$ is projective, for any family I, and that $G - \mathrm{Mod}$ is equivalent to $\mathrm{Mod}(\mathbb{Z}[G])$.

Indeed a structure of G-module on an abelian group A extends by linearity as a structure of left module over $\mathbb{Z}[G]$, by setting $(\sum n_g e_g)x := \sum n_g(g\cdot x)$, for $x \in A$. The converse is easy: a $\mathbb{Z}[G]$-module structure on A gives a G-module structure by $g \cdot x := e_g x$.

Now, to see that $R^{(I)}$ is projective in $\mathrm{Mod}(R)$, we use

$$\mathrm{Hom}_{\mathrm{Mod}(R)}(R^{(I)}, M) \simeq M^I,$$

for any $M \in \mathrm{Mod}(R)$.

Lemma 6.17. *Let $\mathcal{C}$ be an abelian category. Let A be a (maybe infinite) set.*

Let P_α, $\alpha \in A$, be projective objects. We assume that $P = \bigoplus_{\alpha\in A} P_\alpha$ exists. Then P is projective.

Let I_α, $\alpha \in A$, be injective objects. We assume that $I = \prod_{\alpha\in A} I_\alpha$ exists. Then I is injective.

Proof. This follows from $\mathrm{Hom}_{\mathcal{C}}(\bigoplus_{\alpha\in A} P_\alpha, X) = \prod_{\alpha\in A} \mathrm{Hom}_{\mathcal{C}}(P_\alpha, X)$ and $\mathrm{Hom}_{\mathcal{C}}(X, \prod_{\alpha\in A} I_\alpha) = \prod_{\alpha\in A} \mathrm{Hom}_{\mathcal{C}}(X, I_\alpha)$. □

Lemma 6.18. *An abelian group is injective if and only if it is* divisible *that is, for any $r \neq 0 \in \mathbb{Z}$ and $a \in A$, there exists $b \in A$ such that $a = rb$.*

Sketch of proof. We assume A is divisible and consider an inclusion of abelian groups $M \subset N$ and a morphism $f\colon M \to A$. We choose $x \in N \setminus M$ and prove that f can be extended to the subgroup $M + \langle x\rangle$ of N; then we could conclude by Zorn Lemma (left to the reader). We have the exact sequence $0 \to M \cap \langle x\rangle \to M \oplus \langle x\rangle \to M + \langle x\rangle \to 0$. We have $\langle x\rangle \simeq \mathbb{Z}/n\mathbb{Z}$ and $M \cap \langle x\rangle \simeq \mathbb{Z}/m\mathbb{Z}$ for some $m \mid n$. We assume $n \neq 0$ (the other case is similar and easier); then $y = \frac{n}{m}x$ is a generator of $M \cap \langle x\rangle$.

By hypothesis there exists $a \in A$ such that $f(y) = \frac{n}{m}a$. We remark that $na = mf(y) = f(my) = f(0) = 0$; hence we can define $f'\colon M \oplus \langle x\rangle \to A$, $(z, kx) \mapsto f(z) - ka$. Then $f'|_{M\cap\langle x\rangle} = 0$ and f' factorizes through $f''\colon M + \langle x\rangle \to A$ which is the required extension of f. ☐

Lemma 6.19. *The category* **Ab** *has enough projectives and enough injectives.*

Proof. (i) We know that $\mathbb{Z}$ is projective. For $M \in \mathbf{Ab}$ and $x \in M$ we define $\varphi_x\colon \mathbb{Z} \to M$, $n \mapsto n \cdot x$. Now the sum of all these maps $\mathbb{Z}^{(M)} \to M$, $(n_x)_{x\in M} \mapsto \sum n_x \cdot x$ is surjective (note that the sum is finite). By Lemma 6.17 $\mathbb{Z}^{(M)}$ is projective.

(ii) Let $M \in \mathbf{Ab}$. For any $x \neq 0 \in M$ we can find a morphism $\psi_x\colon M \to \mathbb{Q}/\mathbb{Z}$ such that $\psi_x(x) \neq 0$. Indeed we first define ψ_x on the subgroup $\langle x\rangle \subset M$ by $\psi_x(nx) = [1/m]$, where m is the order of x, if $m \neq \infty$, and $\psi_x(nx) = [n/2]$ if $m = \infty$; then we can extend ψ_x to M since $\mathbb{Q}/\mathbb{Z}$ is injective (by Lemma 6.18). Now we make the product of these maps and define $\psi\colon M \to \mathbb{Q}/\mathbb{Z}^M$, $y \mapsto (\psi_x(y))_{x\in M}$. Then ψ is injective and $\mathbb{Q}/\mathbb{Z}^M$ is injective. ☐

7. DM

Exercice 7.1. On travaille dans une catégorie abélienne. Soit $f\colon X \to Y$, $g\colon Y \to Z$ deux morphismes. On rappelle que $\mathrm{Hom}(X \oplus Y, Z) \simeq \mathrm{Hom}(X, Z) \oplus \mathrm{Hom}(Y, Z)$ et on note (a, b) les morphismes via cet isomorphisme. De même $\mathrm{Hom}(W, X \oplus Y) \simeq \mathrm{Hom}(W, X) \oplus \mathrm{Hom}(W, Y)$ et on note $\binom{c}{d}$ les morphismes. Ainsi $(a, b) \circ \binom{c}{d} = a \circ c + b \circ d$.

1) Montrer qu'on a une suite exacte $0 \to X \xrightarrow{u} X \oplus Y \xrightarrow{v} Y \to 0$ où $u = \binom{\mathrm{id}_X}{f}$, $v = (f, -\mathrm{id}_Y)$.

2) Montrer qu'on a une suite exacte $0 \to \ker f \to \ker g \circ f \to \ker g \to \operatorname{coker} f \to \operatorname{coker} g \circ f \to \operatorname{coker} g \to 0$.

Exercice 7.2. On considère le diagramme commutatif suivant dans une catégorie abélienne:

$$\begin{array}{ccccccc} A & \longrightarrow & B & \xrightarrow{f} & C & \xrightarrow{g} & D \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} \\ A' & \xrightarrow{h} & B' & \longrightarrow & C' & \longrightarrow & D'. \end{array}$$

On suppose que les lignes sont exactes, que a est un épimorphisme et b, d des monomorphismes. On veut montrer que c est un monomorphisme en utilisant le lemme du serpent. Soit $x\colon X \to C$ tel que $c \circ x = 0$.

Montrer que x factorise par $y\colon X \to \operatorname{im} f$.

Montrer qu'on a le diagramme commutatif

$$\begin{array}{ccccccccc} & & A & \longrightarrow & B & \longrightarrow & \operatorname{im} f & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a_1} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c_1} & & \\ 0 & \longrightarrow & \operatorname{im} h & \xrightarrow{h} & B' & \longrightarrow & C' & & \end{array}$$

où les lignes sont exactes et a_1, c_1 sont induits par a, c. Conclure.

Exercice 7.3. (1) Soit $F\colon \mathcal{C} \to \mathcal{C}'$ un foncteur additif entre catégories abéliennes. On rappelle que F est exact à gauche si, pour toute suite exacte $0 \to A \to B \to C$ dans $\mathcal{C}$, la suite $0 \to F(A) \to F(B) \to F(C)$ est exacte dans $\mathcal{C}'$.

A la place on suppose seulement que, pour toute suite exacte $0 \to A \to B \to C \to 0$ dans $\mathcal{C}$, la suite $0 \to F(A) \to F(B) \to F(C)$ est exacte dans $\mathcal{C}'$. Montrer que F est exact à gauche.

(2) On rappelle que F est exact si, pour toute suite exacte $0 \to A \to B \to C \to 0$ dans $\mathcal{C}$, la suite $0 \to F(A) \to F(B) \to F(C) \to 0$ est exacte dans $\mathcal{C}'$. Montrer alors que, pour toute suite exacte $A \to B \to C$ dans $\mathcal{C}$, la suite $F(A) \to F(B) \to F(C)$ est exacte dans $\mathcal{C}'$.

Exercice 7.4. Soit $p\colon X \to X$ un morphisme dans une catégorie abélienne tel que $p \circ p = p$. Montrer qu'il existe un isomorphisme $X \simeq Y \oplus Z$ tel que $p = i \circ r$ où $i\colon Y \to X$ et $r\colon X \to Y$ sont les morphismes naturels associés à cette décomposition. (Indication: poser $q = \mathrm{id}_X - p$, décomposer $p = i_p \circ \pi_p\colon P \xrightarrow{\pi_p} \operatorname{im} p \xrightarrow{i_p} P$, de même pour q. Dans la catégorie des espaces vectoriels, que seraient Y et Z?)

Exercice 7.5. Soit $F\colon \mathcal{C} \to \mathcal{C}'$ un foncteur additif entre catégories abéliennes. Montrer que (a) $\Rightarrow$ (b) $\Rightarrow$ (c):

(a) F est fidèle, c'est-à-dire, $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(X), F(Y))$ est injectif, pour tout $X, Y \in \mathcal{C}$,
(b) F est conservatif, c'est-à-dire, pour $u\colon X \to Y$ dans $\mathcal{C}$, si $F(u)$ est un isomorphisme, alors u est un isomorphisme,
(c) pour $X \in \mathcal{C}$, si $F(X) \simeq 0$, alors $X \simeq 0$.

Montrer que, si F est exact, alors $(c) \Rightarrow$ (a).

(Indication: pour (a) $\Rightarrow$ (b), soit $j\colon \ker u \to X$, montrer que $F(j) = 0\dots$; pour $(c) \Rightarrow$ (a) avec F exact, on pourra montrer d'abord que F est conservatif.)

Exercice 7.6. Soit X un espace topologique. Montrer que le foncteur $\mathbf{Psh}(X) \to \mathbf{Sh}(X)$, $P \mapsto P^a$ est exact. On rappelle qu'une suite de faisceaux $A \to B \to C$ est exacte si et seulement si les suites de germes $A_x \to B_x \to C_x$ sont exactes pour tout $x \in X$.

Exercice 7.7. Soit X un espace topologique. Pour un faisceau F sur X et un ouvert $U \subset X$ on note $F|_U$ le faisceau sur U défini par $F|_U(V) = F(V)$, $V \subset U$. On obtient en fait un foncteur $\mathbf{Sh}(X) \to \mathbf{Sh}(U)$, $F \mapsto F|_U$ (on a aussi un morphisme $\mathrm{Hom}_{\mathbf{Sh}(X)}(F, G) \to \mathrm{Hom}_{\mathbf{Sh}(U)}(F|_U, G|_U)$).

Soit G un autre faisceau sur X. On note $\mathcal{H}om(F, G)$ le préfaisceau sur X défini par $\mathcal{H}om(F, G)(U) = \mathrm{Hom}_{\mathbf{Sh}(U)}(F|_U, G|_U)$. Montrer que c'est un faisceau.

Exercice 7.8. (**) Soit A un groupe abélien et F le faisceau des fonctions à valeurs dans A sur $\mathbb{R}$ (c'est-à-dire $F(U) = \{f\colon U \to A\}$ - sans aucune condition sur f). Noter que $F \simeq \prod_{x \in \mathbb{R}} A_{\{x\}}$. Comme $A_{\mathbb{R}}$ est le faisceau des fonctions localement constantes on a un morphisme naturel $i\colon A_{\mathbb{R}} \to F$. Vérifier que i est un monomorphisme. On définit $G = \operatorname{coker} i$. On a donc la suite exacte $0 \to A_{\mathbb{R}} \to F \xrightarrow{p} G \to 0$.

(i) Montrer que pour tout interval ouvert I de $\mathbb{R}$ le morphisme $F(I) \to G(I)$ est surjectif.

Voici des étapes possibles. On suppose $I = \left]-1, 1\right[$ et on choisit $s \in G(I)$; on sait que $F_0 \to G_0$ est surjectif, donc on peut choisir

$t^0 \in F_0$ tel que $p_0(t^0) = s_0$. On représente t^0 par $t \in F(]-\varepsilon, \varepsilon[)$. Vérifier que, quitte à prendre $\varepsilon > 0$ plus petit on a $p|_{]-\varepsilon,\varepsilon[}(t) = s|_{]-\varepsilon,\varepsilon[}$.

Pour $\varepsilon \leq x < 1$ soit

$$E(x) = \{t' \in F(]-\varepsilon, x[);\ p|_{]-\varepsilon,x[}(t') = s|_{]-\varepsilon,x[},\ t'|_{]-\varepsilon,\varepsilon[} = t\}.$$

Montrer que $E(x)$ est vide ou un singleton. Montrer que si $E(x) \neq \emptyset$ et $x < 1$, alors il existet $x' > x$ tel que $E(x') \neq \emptyset$. Montrer que si x_i est une suite croissante dans $[\varepsilon, 1[$ de limite x et tous les $E(x_i)$ sont non vides, alors $E(x)$ est non vide.

(ii) Montrer que pour tout ouvert U de $\mathbb{R}$ le morphisme $F(U) \to G(U)$ est surjectif.

(iii) Montrer que F est *flasque*, c'est-à-dire pour tout ouvert U, le morphisme de restriction $F(\mathbb{R}) \to F(U)$ est surjectif. Montrer que G est *flasque*.

(iv) On travaille dans la catégorie des faisceaux d'espaces vectoriels sur un corps $\mathbf{k}$. Dans ce cadre on admet que les faisceaux flasques sont injectifs. Calculer $R^i\Gamma(\mathbb{R}; \mathbf{k}_{\mathbb{R}})$, avec la notation $\Gamma(U; F) = F(U)$ (on sait que $\Gamma(U; -)\colon \mathbf{Sh}(X) \to \mathbf{Ab}$ est exact à gauche et on note $R^i\Gamma(U; -)$ ses foncteurs dérivés à droite – une autre notation standard est $H^i(U; -)$).

8. Adjoint functors

Let $\mathcal{C}$, $\mathcal{C}'$ be categories and let $R\colon \mathcal{C}' \to \mathcal{C}$, $L\colon \mathcal{C} \to \mathcal{C}'$ be two functors. Roughly speaking, we say that L is left adjoint to R if $\mathrm{Hom}_{\mathcal{C}}(X, R(Y)) \simeq \mathrm{Hom}_{\mathcal{C}'}(L(X), Y)$ for all $X \in \mathcal{C}$, $Y \in \mathcal{C}'$. Of course we want these isomorphisms to be functorial in X and Y. For this we remark that $\mathrm{Hom}_{\mathcal{C}}(\cdot, \cdot)$ is a functor from $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ to **Set**. In the same way $\mathrm{Hom}_{\mathcal{C}}(\cdot, R(\cdot))$ and $\mathrm{Hom}_{\mathcal{C}'}(L(\cdot), \cdot)$ are functors from $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}'$ to **Set**. Now we can give a formal definition:

Definition 8.1. Let $\mathcal{C}$, $\mathcal{C}'$ be categories and let $R\colon \mathcal{C}' \to \mathcal{C}$, $L\colon \mathcal{C} \to \mathcal{C}'$ be two functors. We say that L is left adjoint to R (or R right adjoint to L, or (L, R) is an adjoint pair) if there exists an isomorphism of functors from $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}'$ to **Set**:

$$\mathrm{Hom}_{\mathcal{C}}(\cdot, R(\cdot)) \simeq \mathrm{Hom}_{\mathcal{C}'}(L(\cdot), \cdot). \tag{8.1}$$

It is called the adjunction morphism.

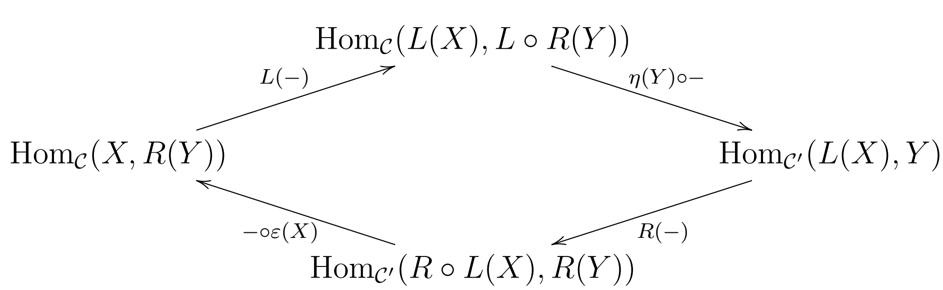
Lemma 8.2. *let $F\colon \mathcal{C} \to \mathcal{C}'$ be a functor. If F has a right (or left) adjoint, then this adjoint is unique, up to a canonical isomorphism.*

Proof. Let G, G' be two right adjoints. For any $X \in \mathcal{C}$, $Y \in \mathcal{C}'$ we have $\mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \simeq \mathrm{Hom}_{\mathcal{C}'}(F(X), Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X, G'(Y))$. Setting $X = G(Y)$ the image of $\mathrm{id}_{G(Y)}$ gives $\theta(Y)\colon G(Y) \to G'(Y)$. Using the functoriality we see that θ is a morphism of functors. Switching G, G' gives $\theta'\colon G' \to G$. By construction the composition $\theta' \circ \theta$ gives the identity morphism $\mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \to \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$, $\forall X, Y$, and it follows that $\theta' \circ \theta = \mathrm{id}$. ☐

Setting $X = R(Y)$ in the equality $\mathrm{Hom}_{\mathcal{C}}(X, R(Y)) \simeq \mathrm{Hom}_{\mathcal{C}'}(L(X), Y)$ the image of $\mathrm{id}_{R(Y)}$ gives $\eta(Y)\colon L \circ R(Y) \to Y$. As in the proof of Lemma 8.2 we can see η is a morphism of functors. Setting $Y = L(X)$ gives a morphism in the other direction. So we obtain

$$\varepsilon\colon \mathrm{id}_{\mathcal{C}} \to R \circ L, \qquad \eta\colon L \circ R \to \mathrm{id}_{\mathcal{C}'} \tag{8.2}$$

and we can check that the bijection (8.1) is given as the compositions



Runing around this diagram gives the identity morphisms at left and right hand sides. Setting $Y = L(X)$ or $X = R(Y)$ we deduce that the following compositions are the identity morphisms:

$$(\eta \circ L) \circ (L \circ \varepsilon) = \mathrm{id}_L \colon L \to L \circ R \circ L \to L, \tag{8.3}$$

$$(R \circ \eta) \circ (\varepsilon \circ R) = \mathrm{id}_R \colon R \to R \circ L \circ R \to R. \tag{8.4}$$

We can prove (see for example [3] Prop. 1.5.4):

Lemma 8.3. *If L, R are functors and ε, η morphisms of functors satisfying* (8.3), (8.4)*, then (L, R) is an adjoint pair.*

Example 8.4. Let $for\colon \mathbf{Ab} \to \mathbf{Set}$ be the forgetful functor. Then for has a left adjoint, the "free abelian group" functor $I \mapsto \mathbb{Z}^{(I)}$, that is, $\mathrm{Hom}_{\mathbf{Set}}(I, for(A)) \simeq \mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}^{(I)}, A)$.

In the same way we can define the functors "free **k**-module" for a ring **k**, "free group", "free associative **k**-algebra",...

Example 8.5. Let X be a topological space. The forgetful functor $for\colon \mathbf{Sh}(X) \to \mathbf{Psh}(X)$ and the "associated sheaf functor" $\mathbf{Psh}(X) \to \mathbf{Sh}(X)$, $P \mapsto P^a$ are adjoint: we have the functorial isomorphism

$$\mathrm{Hom}_{\mathbf{Psh}(X)}(P, for(F)) \simeq \mathrm{Hom}_{\mathbf{Sh}(X)}(P^a, F)$$

for $P \in \mathbf{Psh}(X)$, $F \in \mathbf{Sh}(X)$.

Exercise 8.6. Let $\mathcal{C}$, $\mathcal{C}'$ be abelian categories and let $R\colon \mathcal{C}' \to \mathcal{C}$, $L\colon \mathcal{C} \to \mathcal{C}'$ be additive functors such that R is right adjoint to L. Prove that R is left exact and L is right exact.

9. Direct and inverse images of sheaves

Let $f\colon X \to Y$ be a continuous map between topological spaces. Let $\mathbf{k}$ be a ring. We define the direct image functor $f_*\colon \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ and the inverse image functor $f^{-1}\colon \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$. They form an adjoint pair (f^{-1}, f_*). When X and Y are Hausdorff and locally compact we also define the proper direct image functor $f_!\colon \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$.

Definition 9.1. For $F \in \mathbf{Psh}(X)$ we define $f_*F \in \mathbf{Psh}(Y)$ by its sections $(f_*F)(V) = F(f^{-1}(V))$ for any open subset $V \subset Y$, with the restriction maps naturally given by those of F. If $F \in \mathbf{Sh}(X)$ we can check that $f_*F \in \mathbf{Sh}(Y)$.

If $u\colon F \to G$ is a morphism in $\mathbf{Sh}(X)$, we define $f_*u\colon f_*F \to f_*G$ by $(f_*u)(V) = u(f^{-1}(V))$. We obtain functors $f_*\colon \mathbf{Psh}(X) \to \mathbf{Psh}(Y)$, $f_*\colon \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$.

Lemma 9.2. *The functor $f_*\colon \mathbf{Psh}(X) \to \mathbf{Psh}(Y)$ is exact and the functor $f_*\colon \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ is left exact.*

Definition 9.3. For $G \in \mathbf{Psh}(Y)$ we define a presheaf ${}^{\mathrm{pr}}f^{-1}G \in \mathbf{Psh}_{\mathbf{k}}(X)$ by $({}^{\mathrm{pr}}f^{-1}G)(U) = \varinjlim_{V \supset f(U)} G(V)$, where V runs over the open neighborhoods of $f(U)$ in Y. The restriction maps are naturally induced by those of G. A morphism $u\colon F \to G$ induces morphisms on the inductive limits, $({}^{\mathrm{pr}}f^{-1}u)(U)\colon ({}^{\mathrm{pr}}f^{-1}F)(U) \to ({}^{\mathrm{pr}}f^{-1}G)(U)$, for all $U \in \mathrm{Op}(X)$, which are compatible and define ${}^{\mathrm{pr}}f^{-1}u\colon {}^{\mathrm{pr}}f^{-1}F \to {}^{\mathrm{pr}}f^{-1}G$. This gives a functor ${}^{\mathrm{pr}}f^{-1}\colon \mathbf{Psh}_{\mathbf{k}}(X) \to \mathbf{Psh}_{\mathbf{k}}(Y)$.

We set $f^{-1}G = ({}^{\mathrm{pr}}f^{-1}G)^a$ and obtain a functor $f^{-1}\colon \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$.

When $f\colon X \to Y$ is an embedding of topological spaces (that is, f is an inclusion and the topology of X is the induced topology) we often write

$$G|_X := f^{-1}G.$$

If f is the inclusion of an open set, we have $(G|_X)(U) = G(U)$, for all $U \in \mathrm{Op}(X)$.

Exercise 9.4. Let X be a Hausdorff topological space and $Z \subset X$ a compact subset. Then, for any $F \in \mathbf{Sh}(X)$ and $V \in \mathrm{Op}(Z)$, we have $(F|_Z)(V) \simeq \varinjlim_{U \supset V} F(U)$, where U runs over the open neighborhoods of V in X.

Lemma 9.5. *Let $f\colon X \to Y$ be a continuous map and let $x \in Y$. For any $F \in \mathbf{Sh}(Y)$ we have a natural isomorphism $(f^{-1}F)_x \simeq F_{f(x)}$.*

Since the exactness of a sequence of sheaves can be checked in the stalks we deduce:

Lemma 9.6. *For any continuous map $f\colon X \to Y$, the functor f^{-1} is exact.*

In the situation of Definitions 9.1 and 9.3 we define two morphisms of functors

$$\varepsilon\colon \mathrm{id}_{\mathbf{Psh}(Y)} \to f_* \circ {}^{\mathrm{pr}}f^{-1}, \qquad \eta\colon {}^{\mathrm{pr}}f^{-1} \circ f_* \to \mathrm{id}_{\mathbf{Psh}(X)} \tag{9.1}$$

as follows. For $G \in \mathbf{Psh}(Y)$ and $V \in \mathrm{Op}(Y)$ we have

$$f_* \circ {}^{\mathrm{pr}}f^{-1}(G)(V) = ({}^{\mathrm{pr}}f^{-1}G)(f^{-1}V) = \varinjlim_{W} G(W),$$

where $W \subset Y$ runs over the open subsets such that $f(f^{-1}(V)) \subset W$. We remark that V belongs to this family of W's. Hence we have a natural morphism $G(V) \to f_* \circ {}^{\mathrm{pr}}f^{-1}(G)(V)$. It is compatible with the restrictions maps for $V' \subset V$ and gives $\varepsilon(G)\colon G \to f_* \circ {}^{\mathrm{pr}}f^{-1}(G)$. For any $F \in \mathbf{Sh}(X)$ and $U \in \mathrm{Op}(X)$ we have

$${}^{\mathrm{pr}}f^{-1} \circ f_*(F)(U) = \varinjlim_{W} F(f^{-1}(W)),$$

where $W \subset X$ runs over the open subsets such that $f(U) \subset W$, that is, $U \subset f^{-1}(W)$. We thus have a natural morphism ${}^{\mathrm{pr}}f^{-1} \circ f_*(F)(U) \to F(U)$, which defines our η.

We can check the hypothesis of Lemma 8.3 and deduce that $({}^{\mathrm{pr}}f^{-1}, f_*)$ is an adjoint pair. Using Example 8.5 we obtain:

Proposition 9.7. *Let $f\colon X \to Y$ be a continuous map between topological spaces. The pairs of functors $({}^{\mathrm{pr}}f^{-1}, f_*)$ and (f^{-1}, f_*) are adjoint pairs.*

Definition 9.8. Let $F \in \mathbf{Sh}(X)$, $U \in \mathrm{Op}(X)$ and $s \in F(U)$. The support of s is the closed subset $\mathrm{supp}(s)$ of U defined by

$$U \setminus \mathrm{supp}(s) = \bigcup_{V \in \mathrm{Op}(U),\, s|_V \simeq 0} V.$$

Alternatively $U \setminus \mathrm{supp}(s)$ is the biggest open subset V of U such that $s|_V \simeq 0$.

A topological space X is locally compact if, for any $x \in X$ and any neighborhood U of x, there exists a compact neighborhood of x contained in U. Now we assume X, Y are Hausdorff and locally compact. Then a map $f\colon X \to Y$ is *proper* if the inverse image of any compact subset of Y is compact.

Definition 9.9. Let $f\colon X \to Y$ be a continuous map of Hausdorff and locally compact spaces. For $F \in \mathbf{Sh}(X)$ we define a subsheaf $f_!F \in \mathbf{Sh}(Y)$ of f_*F by

$$(f_!F)(V) = \{s \in (f^{-1}(V));\ f|_{\operatorname{supp} s}\colon \operatorname{supp}(s) \to V \quad \text{is proper}\}$$

for any open subset $V \subset Y$. If $u\colon F \to G$ is a morphism in $\mathbf{Sh}(X)$, the morphism $f_*u\colon f_*F \to f_*G$ sends $f_!F$ to $f_!G$. We obtain a functor $f_!\colon \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$.

If the map f itself is proper, then we have $f_! \xrightarrow{\sim} f_*$.

Definition 9.10. Let X be a Hausdorff and locally compact topological space. For $F \in \mathbf{Sh}(X)$ and $U \in \operatorname{Op}(X)$ we set

$$\Gamma_c(U; F) = \{s \in F(V);\ \operatorname{supp}(s) \text{ is compact.}\}$$

Proposition 9.11. *Let $f\colon X \to Y$ be as in Definition 9.9. For any $F \in \mathbf{Sh}(X)$ and $y \in Y$ we have*

$$(f_!F)_y \simeq \Gamma_c(f^{-1}(y); F|_{f^{-1}(y)}).$$

10. More general resolutions

Let $\mathcal{C}$, $\mathcal{C}'$ be abelian categories. We assume that $\mathcal{C}$ has enough injectives. For a given left exact functor $F\colon \mathcal{C} \to \mathcal{C}'$ we may compute RF with more general resolutions than injective resolutions.

Definition 10.1. An object $X \in \mathcal{C}$ such that $R^iF(X) \simeq 0$ for all $i \geq 1$ is called *F-acyclic.*

Lemma 10.2. *Let $0 \to X^0 \to X^1 \to \cdots \to X^n \to 0$ be an exact sequence in $\mathcal{C}$. We assume that $X^0, \dots, X^{n-1}$ are F-acyclic. Then X^n is F-acyclic.*

Proof. We proceed by induction on n. The case $n = 2$ is true by the long exact sequence $\cdots \to R^iF(X^1) \to R^iF(X^2) \to R^{i+1}F(X^0) \to \cdots$.

For $n > 2$ we split our sequence in two exact sequences

$$0 \to X^0 \to X^1 \to \cdots \to X^{n-2} \to Y^{n-1} \to 0,$$
$$0 \to Y^{n-1} \to X^{n-1} \to X^n \to 0,$$

where $Y^{n-1} = \operatorname{im}(X^{n-2} \to X^{n-1}) \simeq \ker(X^{n-1} \to X^n)$. By induction Y^{n-1} is F-acyclic. Hence X^n is F-acyclic by the case $n = 2$. $\square$

Lemma 10.3. *Let $0 \to X^0 \to X^1 \to \cdots \to X^n \to 0$ be an exact sequence in $\mathcal{C}$ (here n may be ∞). We assume that $X^0, \dots, X^n$ are F-acyclic. Then the sequence $0 \to F(X^0) \to F(X^1) \to \cdots \to F(X^n) \to 0$ is exact.*

Proof. We proceed as in the proof of the previous lemma. The case $n = 2$ is true since $R^1F(X^0) \simeq 0$. In general we split the sequence as in the previous lemma. Then Y^{n-1} is F-acyclic and the induction gives the exact sequences

$$0 \to F(X^0) \to F(X^1) \to \cdots \to F(X^{n-2}) \to F(Y^{n-1}) \to 0,$$
$$0 \to F(Y^{n-1}) \to F(X^{n-1}) \to F(X^n) \to 0,$$

which glue into the exact sequence of the current lemma. $\square$

Proposition 10.4. *Let $X \in \operatorname{Ob}(\mathcal{C})$ and let $J \in \mathbf{C}^+(\mathcal{J})$ be a resolution of X by F-acyclic objects. Then we have an isomorphism $R^iF(X) \simeq H^i\,\mathbf{C}(F)(J)$.*

Proof. (i) As in Proposition 6.3, we can find an injective resolution $I \in \mathbf{C}^+(\mathcal{C})$ of X and a morphism $u\colon J \to I$ in $\mathbf{C}(\mathcal{C})$ which lifts id_X:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & X & \xrightarrow{\varepsilon^0} & J^0 & \xrightarrow{d_J^0} & J^1 & \longrightarrow & J^2 & \longrightarrow & \cdots \\
 & & \| & & \downarrow{\scriptstyle u^0} & & \downarrow{\scriptstyle u^1} & & \downarrow{\scriptstyle u^2} & & \\
0 & \longrightarrow & X & \xrightarrow{\varepsilon^1} & I^0 & \xrightarrow{d_I^0} & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots
\end{array}$$

and such that the u^k are monomorphisms. Indeed we first choose a monomorphism u^0 with I^0 injective and set $\varepsilon^1 = u^0 \circ \varepsilon^0$. This gives the first commutative square. We define $p^0\colon I^0 \to \operatorname{coker}(\varepsilon^1)$ and $v^0 = p^0 \circ u^0$. We set $J'^1 = \operatorname{coker}\binom{-v^0}{d_J^0}$ so that we have the commutative diagram

$$\begin{array}{ccc} J^0 & \xrightarrow{d_J^0} & J^1 \\ \downarrow{\scriptstyle v^0} & & \downarrow{\scriptstyle w^1} \\ \operatorname{coker}(\varepsilon^1) & \xrightarrow{i^0} & J'^1 \end{array}$$

where i_0 is a monomorphism (check!). We choose a monomorphism $j^1\colon J'^1 \to J^1$ with J^1 injective and set $d_I^0 = j^1 \circ i^0 \circ p^0$, $u^1 = j^1 \circ w^1$. Since $j^1 \circ i^0$ is a monomorphism, we have $\ker(d_I^0) = \ker(p^0) = \operatorname{im}(\varepsilon^1)$. This gives the second commutative square. We go on by induction.

(ii) We set $K = \operatorname{coker}(u)$. Then $0 \to J \to I \to K \to 0$ is a short exact sequence and Proposition 3.9 implies that $H^iK \simeq 0$ for all $i \in \mathbb{Z}$. Hence, viewing K as a long sequence in $\mathcal{C}$, it is an exact long sequence (we say that K is an acyclic complex).

By Lemma 10.2 with $n = 2$, the K^i's are F-acyclic. By Lemma 10.3 we deduce that the long sequence $F(K^\cdot)$ is exact. In other words $H^i(\mathbf{C}(F)(K)) \simeq 0$ for all $i \in \mathbb{Z}$.

By Lemma 10.3 again, with $n = 2$, the sequences $0 \to F(J^i) \to F(I^i) \to F(K^i) \to 0$ are exact. Now the result follows from Proposition 3.9. □

Definition 10.5. A family of objects $\mathcal{J} \subset \operatorname{Ob}(\mathcal{C})$ is called F-injective if

(i) for any $X \in \operatorname{Ob}(\mathcal{C})$ there exist $J \in \mathcal{J}$ and a monomorphism $0 \to X \to J$,
(ii) for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$, if $X' \in \mathcal{J}$ and $X \in \mathcal{J}$, then $X'' \in \mathcal{J}$,
(iii) for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$ with $X', X, X'' \in \mathcal{J}$, the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ is exact.

Lemma 10.6. *The objects in $\mathcal{J}$ are F-acyclic.*

Proof. We choose an injective object $I \in \mathcal{C}$ and a monomorphism $a\colon J \to I$. Then we choose $J' \in \mathcal{J}$ and a monomorphism $b\colon I \to J'$.

We set $X = \operatorname{coker}(a)$ and $J'' = \operatorname{coker}(b)$. We have the exact sequences

$$\begin{array}{ccccccccc} 0 & \to & J & \to & I & \to & X & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & J & \to & J' & \to & J'' & \to & 0 \end{array}$$

and the long cohomology exact sequences

$$\begin{array}{ccccccccc} 0 & \to & F(J) & \to & F(I) & \to & F(X) & \to & R^1F(J) & \to & R^1F(I) \\ & & \| & & \downarrow & & \downarrow & & \| & & \downarrow \\ 0 & \to & F(J) & \to & F(J') & \to & F(J'') & \xrightarrow{u} & R^1F(J) & \xrightarrow{v} & R^1F(J'), \end{array}$$

$$\begin{array}{ccccc} R^iF(X) & \to & R^{i+1}F(J) & \to & R^{i+1}F(I) \\ \downarrow & & \| & & \downarrow \\ R^iF(J'') & \to & R^{i+1}F(J) & \xrightarrow{v^{i+1}} & R^{i+1}F(J'), \end{array}$$

Now we prove the result by induction on i.

We first consider $i = 1$. By (ii) of Definition 10.5 we have $J'' \in \mathcal{J}$ and then (iii) implies that the morphism u is zero. Hence v is a monomorphism. Since v factorizes through $R^1F(I)$, which is zero since I is injective, we deduce that $R^1F(J)$ is zero, as claimed.

Assuming the result true for i, we have $R^iF(J'') \simeq 0$ since $J'' \in \mathcal{J}$. Hence the morphism v^{i+1} is a monomorphism and we conclude as in the case $i = 1$ that $R^{i+1}F(J) \simeq 0$. □

We can modify the proof of Prop. 6.3 to obtain:

Proposition 10.7. *If $\mathcal{J}$ is an F-injective family, then any complex $X \in \mathbf{C}^+(\mathcal{C})$ has a resolution by objects of $\mathcal{J}$, that is, there exist a complex of objects in $\mathcal{J}$, $J \in \mathbf{C}^+(\mathcal{J})$, and a morphism $u\colon X \to J$ in $\mathbf{C}^+(\mathcal{C})$ such that u is a qis.*

Flabby and soft sheaves. Let X be a topological space.

Definition 10.8. A sheaf $F \in \mathbf{Sh}(X)$ is flabby if, for any open subset $U \subset X$, the restriction morphism $F(X) \to F(U)$ is surjective.

Let $f\colon X \to Y$ be a continuous map.

Proposition 10.9 (see [2], Section 2.4)**.** *The family of flabby sheaves is f_*-injective and $f_!$-injective.*

We apply this proposition to the computation of $H^i(\mathbb{R}; \mathbf{k}_{[a,b]})$ for a closed interval $[a, b]$ of $\mathbb{R}$.

We recall the morphism (8.2) $\varepsilon\colon \mathbf{k}_{[a,b]} \to i_* i^{-1}\mathbf{k}_{[a,b]}$, where $i\colon \mathbb{R}_{disc} \to \mathbb{R}$ is the map from $\mathbb{R}$ with the discrete topology to $\mathbb{R}$. We can identify $i_* i^{-1}\mathbf{k}_{[a,b]}$ with the sheaf $\mathcal{F}_{[a,b]}$ of functions on $[a,b]$ defined by $\mathcal{F}_{[a,b]}(U) = \{f\colon U \cap [a,b] \to \mathbb{R}\}$. This sheaf is flabby since we can extend a function defined on $U \cap [a,b]$ arbitrarily to a function defined on $[a,b]$. We define $G = \operatorname{coker}(\varepsilon)$ and we have the short exact sequence:

$$0 \to \mathbf{k}_{[a,b]} \to \mathcal{F}_{[a,b]} \to G \to 0. \tag{10.1}$$

Lemma 10.10. *For any open subset $U \subset \mathbb{R}$ the sequence* (10.1) *gives the exact sequence of sections:*

$$0 \to \Gamma(U; \mathbf{k}_{[a,b]}) \xrightarrow{a(U)} \Gamma(U; \mathcal{F}_{[a,b]}) \xrightarrow{b(U)} \Gamma(U; G) \to 0. \tag{10.2}$$

Proof. (i) Writing U as a disjoint union of intervals, $U = \bigsqcup_{k\in\mathbb{Z}} I_k$, we have $\Gamma(U;F) \simeq \prod_{k\in\mathbb{Z}} \Gamma(I_k;F)$. Since a product of exact sequences of abelian groups is exact, we can assume that U is an interval. We have to check that the last morphism in (10.2) is surjective. Let $s \in \Gamma(U;G)$ be given.

(ii) Let us first prove that for any compact subinterval $K = [c,d] \subset U$ there exists a neighborhood V of K and $s' \in \Gamma(V; \mathcal{F}_{[a,b]})$ such that $b(V)(s') = s|_V$.

For any $x \in U$ there exist a neighborhood W_x of x and $s'(x) \in \Gamma(W_x; \mathcal{F}_{[a,b]})$ such that $b(W_x)(s'(x)) = s|_{W_x}$. We can assume that the W_x are intervals and we choose a finite number of them to cover K. We denote them $W_1, \dots, W_N$ and order them so that $V_i := W_1 \cup \dots \cup W_i$ is connected, for all i. We also write $s'_i \in \Gamma(W_i; \mathcal{F}_{[a,b]})$ instead of $s'(x)$. Let us prove by induction on i that there exists $s''_i \in \Gamma(V_i; \mathcal{F}_{[a,b]})$ such that $b(V_i)(s''_i) = s|_{V_i}$. For $i = 1$ we have $s''_1 = s'_1$. Assuming s''_i is defined we have

$$b(V_i \cap W_{i+1})(s''_i|_{V_i\cap W_{i+1}}) = s|_{V_i\cap W_{i+1}} = b(V_i \cap W_{i+1})(s'_{i+1}|_{V_i\cap W_{i+1}}).$$

Hence there exists $t_i \in \Gamma(V_i \cap W_{i+1}; \mathbf{k}_{[a,b]})$ such that

$$s''_i|_{V_i\cap W_{i+1}} - s'_{i+1}|_{V_i\cap W_{i+1}} = a(V_i \cap W_{i+1})(t_i).$$

We can extend t_i to a section $t'_i \in \Gamma(W_{i+1}; \mathbf{k}_{[a,b]})$ because $V_i \cap W_{i+1}$ is connected. We set $\tilde{s}_{i+1} = s'_{i+1} - a(W_{i+1})(t'_i)$. Then we see that $s''_i|_{V_i\cap W_{i+1}} = \tilde{s}_i|_{V_i\cap W_{i+1}}$ and we can glue these sections in a section s''_{i+1} such that $b(V_{i+1})(s''_{i+1}) = s|_{V_{i+1}}$.

(iii) We remark that the section $s' \in \Gamma(V; \mathcal{F}_{[a,b]})$ found in (ii) is unique up to the addition of a section of $\Gamma(V; \mathbf{k}_{[a,b]})$, that is, up to the addition of a constant function. Hence, for a given $x_0 \in U$ with $x_0 \in K$, there exists a unique $s' \in \Gamma(V; \mathcal{F}_{[a,b]})$ such that $b(V)(s') = s|_V$ and $s'(x_0) = 0$.

Now we consider an increasing sequence $K_i \subset K_{i+1} \subset U$, $i \in \mathbb{N}$, of compact intervals whose union is U. For each i we have a unique $s'_i \in \Gamma(V_i; \mathcal{F}_{[a,b]})$ such that $b(V_i)(s'_i) = s|_{V_i}$ and $s'_i(x_0) = 0$, where V_i is some neighborhood of K_i. By unicity we have $s'_{i+1}|_{V_i} = s'_i$. Hence we can glue the s'_i in a section $s' \in \Gamma(U; \mathcal{F}_{[a,b]})$ such that $b(U)(s') = s$. □

Lemma 10.11. *The sheaf G of* (10.1) *is flabby.*

Proof. Let $U \subset \mathbb{R}$ and $s \in G(U)$ be given. By Lemma 10.10 there exists $s' \in \mathcal{F}_{[a,b]}(U)$ such that $b(U)(s') = s$. Since $\mathcal{F}_{[a,b]}$ is flabby, there exists $t' \in \mathcal{F}_{[a,b]}(\mathbb{R})$ such that $t'|_U = s'$. Then $t = b(\mathbb{R})(t')$ satisfies $t|_U = s$. □

Hence (10.1) gives a flabby resolution of $\mathbf{k}_{[a,b]}$. We deduce that for any open subset U of $\mathbb{R}$

$$H^i(U; \mathbf{k}_{[a,b]}) \simeq H^i(0 \to \Gamma(U; \mathcal{F}_{[a,b]}) \xrightarrow{b(U)} \Gamma(U; G) \to 0).$$

By Lemma 10.10 the morphism $b(U)$ is surjective and we obtain that the cohomology of $\mathbf{k}_{[a,b]}$ is concentrated in degree 0:

Proposition 10.12. *Let $[a,b]$ be a closed interval in $\mathbb{R}$. For any open interval U of $\mathbb{R}$ such that $U \cap [a,b] \neq \emptyset$, we have*

$$H^0(U; \mathbf{k}_{[a,b]}) \simeq \mathbf{k} \qquad \textit{and} \qquad H^i(U; \mathbf{k}_{[a,b]}) \simeq 0 \quad \textit{for } i \neq 0.$$

Now we assume X is Hausdorff and locally compact.

Definition 10.13. A sheaf $F \in \mathbf{Sh}(X)$ is c-soft if, for any compact subset $C \subset X$, the restriction morphism $F(X) \to F(C)$ is surjective, where $F(C) := \varinjlim_{C \subset U} F(U)$, U running over the open neighborhoods of C.

We remark that a flabby sheaf is c-soft.

An important example is the case where X is a C^∞ manifold and $F = C^\infty_X$ is the sheaf of C^∞ functions on X. More generally any sheaf of modules over C^∞_X is c-soft, in particular the sheaf of i-forms Ω^i_X is c-soft.

Let $f \colon X \to Y$ be a continuous map, X, Y Hausdorff and locally compact.

Proposition 10.14 (see [2], Section 2.5)**.** *The family of c-soft sheaves is f_*-injective and $f_!$-injective.*

Corollary 10.15. *Let X be a C^∞ manifold. Then $H^i(X; \mathbb{R}_{\mathbb{R}^n}) \simeq H^i_{dR}(X)$, where $H^i_{dR}(X)$ is the de Rham cohomology of X.*

Proof. By the Poincaré lemma the de Rham complex $0 \to \Omega^0_X \to \Omega^1_X \to \cdots \to \Omega^n_X \to 0$, $n = \dim X$, is a c-soft resolution of $\mathbb{R}_X$. The result follows. □

11. Definition of derived categories

In this section we give a first introduction to derived categories. We only give a brief account on the subject and refer to the first chapter of [2] (or Chapters 10-13 of [3]) for details and proofs.

Definition 11.1. Let $\mathcal{C}$ be an abelian category and let $u\colon X \to Y$ be a morphism in $\mathbf{C}(\mathcal{C})$ or in $\mathbf{K}(\mathcal{C})$. We say that u is a quasi-isomorphism (qis for short) if the morphisms $H^i(u)\colon H^i(X) \to H^i(Y)$ are isomorphisms, for all $i \in \mathbb{Z}$.

A related notion is that of acyclic complexes: a complex X in $\mathbf{C}(\mathcal{C})$ or in $\mathbf{K}(\mathcal{C})$ is *acyclic* (or *exact*) if $H^i(X) \simeq 0$ for all $i \in \mathbb{Z}$ (in other words the long sequence $\cdots X^i \xrightarrow{d^i} X^{i+1} \cdots$ is exact).

Exercise 11.2. Let $u\colon X \to Y$ be a morphism in $\mathbf{C}(\mathcal{C})$. Then u is a qis if and only if $\ker(u)$ and $\operatorname{coker}(u)$ are acyclic.

The derived category of $\mathcal{C}$, denoted $\mathbf{D}(\mathcal{C})$, is obtained from $\mathbf{C}(\mathcal{C})$ by inverting the qis. This process is called *localization*.

Definition 11.3. Let $\mathcal{A}$ be a category and $\mathcal{S}$ a family of morphisms in $\mathcal{A}$. A localization of $\mathcal{A}$ by $\mathcal{S}$ is a category $\mathcal{A}_{\mathcal{S}}$ (a priori in a bigger universe) and a functor $Q\colon \mathcal{A} \to \mathcal{A}_{\mathcal{S}}$ such that

(i) for all $s \in \mathcal{S}$, $Q(s)$ is an isomorphism,
(ii) for any category $\mathcal{B}$ and any functor $F\colon \mathcal{A} \to \mathcal{B}$ such that $F(s)$ is an isomorphism for all $s \in \mathcal{S}$, there exists a functor $F_{\mathcal{S}}\colon \mathcal{A}_{\mathcal{S}} \to \mathcal{B}$ such that $F \simeq F_{\mathcal{S}} \circ Q$,
(iii) denoting by $\mathbf{Func}(\cdot,\cdot)$ the category of functors, the functor $\circ Q\colon \mathbf{Func}(\mathcal{A}_{\mathcal{S}},\mathcal{B}) \to \mathbf{Func}(\mathcal{A},\mathcal{B})$ is fully faithful (which implies unicity of $F_{\mathcal{S}}$ in (ii)).

It is possible to construct $\mathcal{A}_{\mathcal{S}}$ as a category with the same objects as $\mathcal{A}$ and with morphisms defined as chains $(s_1, u_1, s_2, u_2, \ldots, s_n, u_n)$ with $s_i \in \mathcal{S}$ and u_i any morphism in $\mathcal{A}$ modulo some equivalence relation. Such a chain is meant to represent $u_n \circ s_n^{-1} \circ u_{n-1} \circ \cdots \circ s_1^{-1}$. The equivalence relation is generated by $(s_1, u_1, \ldots, s_n, u_n) \sim (s_1, u_1, \ldots, s, s, \ldots, s_n, u_n)$ where (s,s), $s \in \mathcal{S}$, is inserted between u_i and s_{i+1}. The composition is the concatenation.

However in our situation the localization will be obtained by a calculus of fractions and we only need chains of (s,u) length 2. We will not use this fact and refer to Section 11.2.

Definition 11.4. Let $\mathcal{C}$ be an abelian category. The derived category of $\mathcal{C}$ is the localization $\mathbf{D}(\mathcal{C}) = (\mathbf{K}(\mathcal{C}))_{Qis}$. We denote by $Q_{\mathcal{C}}\colon \mathbf{K}(\mathcal{C}) \to \mathbf{D}(\mathcal{C})$ the localization functor (or its composition with $\mathbf{C}(\mathcal{C}) \to \mathbf{K}(\mathcal{C})$).

Starting with $\mathbf{K}^*(\mathcal{C})$ where $* = +, -$ or b, we define in the same way $\mathbf{D}^*(\mathcal{C})$.

The obvious functor $\mathbf{C}(\mathcal{C}) \to \mathbf{K}(\mathcal{C})$ sends qis to qis and hence induces a functor $(\mathbf{C}(\mathcal{C}))_{Qis} \to (\mathbf{K}(\mathcal{C}))_{Qis}$. We can prove that this functor is an equivalence (see [1]). So we could as well define $\mathbf{D}(\mathcal{C})$ directly from $\mathbf{C}(\mathcal{C})$. The point is that, starting from $\mathbf{K}(\mathcal{C})$, the localization can be constructed by a calculus of fractions.

The categories $\mathbf{K}(\mathcal{C})$ and $\mathbf{D}(\mathcal{C})$ are additive. They are not abelian in general.

By definition the cohomology functors $H^i\colon \mathbf{K}(\mathcal{C}) \to \mathcal{C}$, $i \in \mathbb{Z}$, factorize through the localization functor. We still denote by $H^i\colon \mathbf{D}(\mathcal{C}) \to \mathcal{C}$ the induced functors.

Lemma 11.5. *Let $\mathcal{C}, \mathcal{C}'$ be abelian categories. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be an exact functor. Then $C(F)$ sends acyclic complexes to acyclic complexes and it sends qis to qis. In particular $Q_{\mathcal{C}'} \circ \mathbf{K}(F)\colon \mathbf{K}(\mathcal{C}) \to \mathbf{D}(\mathcal{C}')$ sends qis to isomorphisms and factorizes in a unique way through a functor $\mathbf{D}(\mathcal{C}) \to \mathbf{D}(\mathcal{C}')$ that we still denote by F:*

$$\begin{array}{ccc} \mathbf{K}(\mathcal{C}) & \xrightarrow{\mathbf{K}(F)} & \mathbf{K}(\mathcal{C}') \\ {\scriptstyle Q_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle Q_{\mathcal{C}'}} \\ \mathbf{D}(\mathcal{C}) & \xrightarrow{F} & \mathbf{D}(\mathcal{C}'). \end{array}$$

We have a natural embedding of $\mathcal{C}$ in $\mathbf{C}(\mathcal{C})$ which sends $X \in \mathcal{C}$ to the complex $(X^\cdot, d_X^\cdot) = \cdots \to 0 \to X \to 0 \to \cdots$ with $X^0 = X$ and $X^i = 0$ for $i \neq 0$. This induces by composition other functors $\mathcal{C} \to \mathbf{K}(\mathcal{C})$ and $\mathcal{C} \to \mathbf{D}(\mathcal{C})$. We can check that all these functors are fully faithful embeddings of $\mathcal{C}$ in $\mathbf{C}(\mathcal{C})$, $\mathbf{K}(\mathcal{C})$ or $\mathbf{D}(\mathcal{C})$.

We have the following generalization of Proposition 6.10.

Proposition 11.6. *Let $\mathcal{C}$ be an abelian category. We assume that $\mathcal{C}$ has enough projectives and we let $\mathcal{P}$ be its full subcategory of projective objects. We denote by $Q|_{\mathcal{P}}\colon \mathbf{K}^-(\mathcal{P}) \to \mathbf{D}^-(\mathcal{C})$ the functor induced by the quotient functor. Then $Q|_{\mathcal{P}}$ is an equivalence of categories.*

Similarly, if $\mathcal{C}$ has enough injectives and $\mathcal{I}$ is the full subcategory of injective objects, then $Q|_{\mathcal{I}}\colon \mathbf{K}^+(\mathcal{I}) \to \mathbf{D}^+(\mathcal{C})$ is an equivalence.

Definition 11.7. Let $\mathcal{C}, \mathcal{C}'$ be abelian categories. We assume that $\mathcal{C}$ has enough projectives. Let $F\colon \mathcal{C} \to \mathcal{C}'$ (or $F\colon \mathbf{C}^-(\mathcal{C}) \to \mathbf{C}^-(\mathcal{C}')$) be a right exact functor. Let $\mathbf{K}(F)\colon \mathbf{K}^-(\mathcal{P}) \to \mathbf{K}^-(\mathcal{C}')$ be the functor induced by F. We define $LF\colon \mathbf{D}^-(\mathcal{C}) \to \mathbf{D}^-(\mathcal{C}')$ by $LF = Q_{\mathcal{C}'} \circ \mathbf{K}(F) \circ \mathbf{res}_{proj}$, where $\mathbf{res}_{proj}$ is an inverse to the equivalence $Q|_{\mathcal{P}}$ of Proposition 11.6.

In the same way, if $\mathcal{C}$ has enough injectives and F is left exact, we define $RF\colon \mathbf{D}^+(\mathcal{C}) \to \mathbf{D}^+(\mathcal{C}')$ by $RF = Q_{\mathcal{C}'} \circ \mathbf{K}(F) \circ \mathbf{res}_{inj}$, with $\mathbf{res}_{inj}$ inverse of $Q|_{\mathcal{I}}$.

By definition we have $H^i LF = L^i F$.

If F is exact then $LF \simeq F \simeq RF$ with the notation of Lemma 11.5.

The first interest of introducing the derived category is the possibility to compose derived functors:

Proposition 11.8. *Let $F\colon \mathcal{C} \to \mathcal{C}'$, $G\colon \mathcal{C}' \to \mathcal{C}''$ be left exact functors between abelian categories. We assume that $\mathcal{C}$ and $\mathcal{C}'$ have enough injectives and that F sends the injective objects of $\mathcal{C}$ to G-acyclic objects of $\mathcal{C}'$. Then $R(G \circ F) \simeq RG \circ RF$.*

For example we can compute $H^i_c(\mathbb{R}^n; \mathbf{k}_{\mathbb{R}^n})$ by induction on n in Proposition 11.13 below.

Lemma 11.9. *Let $f\colon X \to Y$, $g\colon Y \to Z$, be continuous maps. Then $R(g \circ f)_* \simeq Rf_* \circ Rg_*$. If the spaces are Hausdorff and locally compact spaces, we also have $R(g \circ f)_! \simeq Rf_! \circ Rg_!$.*

Proof. If $F \in \mathbf{Sh}(X)$ is injective, then $f_*(F)$ is injective. Similarly, if $F \in \mathbf{Sh}(X)$ is c-soft, then $f_!(F)$ is c-soft. Hence we can apply Proposition 11.8 in both cases. ☐

Notation 11.10. For a complex $X = (X^\cdot, d^\cdot_X)$ in $\mathbf{C}(\mathcal{C})$ (or in $\mathbf{K}(\mathcal{C})$ or $\mathbf{D}(\mathcal{C})$) and for $k \in \mathbb{Z}$, we denote by $X[k]$ the shifted complex defined by $(X[k])^i = X^{i+k}$ and $d^i_{X[k]} = (-1)^k d^{i+k}_X$.

In particular, for $X \in \mathcal{C}$ viewed as a complex concentrated in degree 0, the complex $X[k]$ is concentrated in degree $-k$.

Definition 11.11. Let $f\colon X \to Y$ be a continuous map of Hausdorff and locally compact spaces. We say that $F \in \mathbf{Sh}(X)$ is f-soft if, for any $y \in Y$, $F|_{f^{-1}(y)}$ is c-soft.

Proposition 11.12. *The family of f-soft sheaves if $f_!$-injective.*

Proposition 11.13. *Let $p\colon \mathbb{R}^{n+d} \to \mathbb{R}^d$ be the projection and let $\mathbf{k}$ be an abelian group. Then $Rp_!(\mathbf{k}_{\mathbb{R}^{n+d}}) \simeq \mathbf{k}_{\mathbb{R}^d}[-n]$. In particular*

$$H^i_c(\mathbb{R}^n; \mathbf{k}_{\mathbb{R}^n}) \simeq \begin{cases} \mathbf{k} & \text{for } i = n, \\ 0 & \text{for } i \neq n. \end{cases}$$

Proof. If $d > 1$ we can write $p = q \circ r$, with $r\colon \mathbb{R}^{n+d} \to \mathbb{R}^{n-1+d}$ and $q\colon \mathbb{R}^{n-1+d} \to \mathbb{R}^d$. By Lemma 11.9 we can prove the result by induction on n, once we have check the case $n = 1$.

In dimension 1 we have seen that (10.1) gives a flabby, hence c-soft, resolution of $\mathbf{k}_{\mathbb{R}}$. On $\mathbb{R}^{d+1}$ we define $\mathcal{F}$ by $\mathcal{F}(U) = \{f \colon U \to \mathbf{k};\ f|_{U\cap(\mathbb{R}^d\times\{y\})}$ is locally constant, for each $y \in \mathbb{R}\}$. We remark that $\mathbf{k}_{\mathbb{R}^{d+1}}$ is a subsheaf of $\mathcal{F}$ and we define $G = \operatorname{coker}(\mathbf{k}_{\mathbb{R}^{d+1}} \to \mathcal{F})$. By definition we have the exact sequence $0 \to \mathbf{k}_{\mathbb{R}^{d+1}} \to \mathcal{F} \to G \to 0$. For each $x \in \mathbb{R}^d$, its restriction to $r^{-1}(x)$ is (10.1), where $r\colon \mathbb{R}^{d+1} \to \mathbb{R}^d$ is the projection. Hence $\mathcal{F}$ and G are r-soft and we can use the resolution to compute $Rr_!(\mathbf{k}_{\mathbb{R}^{d+1}})$. □

We have used a slight generalization of Lemma 10.10 in the proof of Proposition 11.13 and we generalize a bit more in the next lemma. Let $p\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection and, for each $x \in \mathbb{R}$, let $i_x\colon \mathbb{R}^n \times \{x\}$ be the inclusion. For any $F \in \mathbf{Sh}(\mathbb{R}^{n+1})$ we set

$$R_0(F) = \prod_{x\in\mathbb{R}} i_{x*}i_x^{-1}(F).$$

The adjunctions (i_x^{-1}, i_{x*}) give the natural morphisms $F \to i_{x*}i_x^{-1}(F)$. Since $\operatorname{Hom}(F, R_0(F)) \simeq \prod_{x\in\mathbb{R}} i_{x*}\operatorname{Hom}(F, i_x^{-1}(F))$ we obtain a morphism $\varepsilon(F)\colon F \to R_0(F)$. We set $R_1(F) = \operatorname{coker}(\varepsilon(F))$ and get a sequence

$$0 \to F \xrightarrow{\varepsilon(F)} R_0(F) \to R_1(F) \to 0. \tag{11.1}$$

Lemma 11.14. *Let $p\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection and let $F \in \mathbf{Sh}(\mathbb{R}^{n+1})$. Then*

1. *The morphism $\varepsilon(F)$ is a monomorphism and the sequence* (11.1) *is exact.*
2. *The sheaves $R_0(F)$ and $R_1(F)$ are p-soft.*

Proof. For $y \in \mathbb{R}^n$ we let $j_y\colon \mathbb{R} \to \mathbb{R}^{n+1}$ be the inclusion $x \mapsto (x, y)$. We have $(j_y^{-1}(F))_x \simeq F_{(x,y)}$. Since exactness can be checked at the germs, we can as well restrict first to $\mathbb{R}$ through j_y^{-1}. In the same way, a sheaf G is p-soft if, for each $y \in \mathbb{R}^n$ the sheaf $j_y^{-1}(G)$ is c-soft. Hence we may assume from the beginning that $n = 0$ and we work on $\mathbb{R}$.

(a) For $U \subset \mathbb{R}$ open the morphism $F(U) \to (\prod_{x\in\mathbb{R}}(F_x)_{\{x\}})(U) = \prod_{x\in\mathbb{R}}(F_x)_{\{x\}}(U) = \prod_{x\in U} F_x$ maps a section s to the product of its germs; if the image is 0, then $s_x = 0$ for all $x \in U$ and $s = 0$. Hence $\varepsilon(F)(U)$ is injective.

(b) It is not difficult to see that $R_0(F)$ is flabby, hence c-soft. Let $C \subset \mathbb{R}$ be compact subset and $s \in R_1(F)(C) = \varinjlim_{C\subset U} R_1(F)(U)$, where U is open, be given. We pick U such that s is defined on U.

For each $x \in \overline{C}$ we can choose an open interval $I(x)$ and $t(x) \in R_0(F)(I(x))$ such that $d(t(x)) = s|_{I(x)}$, where $d\colon R_0(F) \to R_1(F)$ is the quotient map. We cover $\overline{C}$ by a finite number of such intervals say $I(x_1), \dots, I(x_N)$. We write $I(x_k) =]a_k, b_k[$. We can assume that the $I(k)$'s are ordered in the sense that $a_k < a_{k+1}$, $b_k < b_{k+1}$. We set $V = \bigcup_{k=1}^{N} I(x_k)$. We first assume for simplicity that V is connected so $V =]a_1, b_N[$ and we set $W =]a_2, b_{N-1}[$.

Since $R_0(F)$ is c-soft we can find another section $u(x_1) \in R_0(F)(I(x_1))$ which coincides with $t(x_1)$ near $I(x_1) \cap (C \cup W)$ and which is 0 near a_1. We set $s_1' = d(u(x_1))$. Then $s_1'|_{I(x_1)\cap W}$ coincides with $s|_{I(x_1)\cap W}$. In the same way we can find $s_N' \in R_1(F)(I(x_N))$ such that $s_N'|_{I(x_N)\cap(C\cup W)}$ coincides with $s|_{I(x_N)\cap(C\cup W)}$ and is 0 near b_N. Then s_1', $s|_W$ and s_N' glue into a section s' of $R_1(F)(V)$ which coincides with s near C and is 0 near a_1 and b_N. Now s' can be extended by 0 on $\mathbb{R}$.

When U has several components we argue in the same way near each component and make the sum of the sections. This gives an extension of s to $\mathbb{R}$ and proves that $R_1(F)$ is c-soft. □

11.1. **Resolutions via double complexes.** Let $\mathcal{C}$ be an abelian category and $X = (\cdots \to 0 \to X^i \to X^{i+1} \to \cdots \to X^j \to 0 \to \cdots)$ be an object of $\mathbf{C}(\mathcal{C})$. We assume that we have a commutative diagram

$$\begin{array}{ccccccc}
X^i & \longrightarrow & X^{i+1} & \longrightarrow & \cdots & \longrightarrow & X^j \\
\downarrow & & \downarrow & & & & \downarrow \\
Y^{i,0} & \xrightarrow{d_1^{i,0}} & Y^{i+1,0} & \xrightarrow{d_1^{i+1,0}} & \cdots & \longrightarrow & Y^{j,0} \\
\downarrow{\scriptstyle d_2^{i,0}} & & \downarrow & & & & \downarrow \\
Y^{i,1} & \longrightarrow & Y^{i+1,1} & \longrightarrow & \cdots & \longrightarrow & Y^{j,1} \\
\downarrow{\scriptstyle d_2^{i,1}} & & \downarrow & & & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

where

- the rows $Y^{i,k} \to Y^{i+1,k} \to \cdots \to Y^{j,k}$ are complexes, for each k,
- the columns $X^l \to Y^{l,0} \to Y^{l,1} \to Y^{l,2} \to \cdots$ are resolutions of X^l, for each l.

We define the *total complex* of $Y^{*,*}$ as the complex $\mathrm{Tot}^*(Y)$ where $\mathrm{Tot}^n(Y) = \bigoplus_{p+q=n} Y^{p,q}$ with differential $d^n = \sum_{p+q=n}(d_1^{p,q} + (-1)^p d_2^{p,q})$. Then we can check:

Lemma 11.15. *The morphisms $X^k \to Y^{k,0}$ define a morphism of complexes $X \to \mathrm{Tot}^*(Y)$ which is a quasi-isomorphism.*

Example 11.16. Let $p\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection and $F = (\cdots \to 0 \to F^i \to F^{i+1} \to \cdots \to F^j \to 0 \to \cdots)$ an object of $\mathbf{C}(\mathbf{Sh}(\mathbb{R}^{n+1}))$.

We use the functors R_0, R_1 of (11.1) in the double complex

$$\begin{array}{ccccccc}
F^i & \longrightarrow & F^{i+1} & \longrightarrow & \cdots & \longrightarrow & F^j \\
\downarrow & & \downarrow & & & & \downarrow \\
R_0(F^i) & \longrightarrow & R_0(F^{i+1}) & \longrightarrow & \cdots & \longrightarrow & R_0(F^j) \\
\downarrow & & \downarrow & & & & \downarrow \\
R_1(F^i) & \longrightarrow & R_1(F^{i+1}) & \longrightarrow & \cdots & \longrightarrow & R_1(F^j)
\end{array}$$

and we deduce a quasi-isomorphism $F \to G$ where $G = 0 \to R_0(F^i) \to (R_0(F^{i+1}) \oplus R_1(F^i)) \to \cdots \to (R_0(F^j) \oplus R_1(F^{j-1}))R_1(F^j) \to 0$. By Lemma 11.14 G is formed by p-soft sheaves.

Using this example and proceeding as in the proof of Proposition 11.13 we can prove

Proposition 11.17. *Let $p\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection and let $F \in \mathbf{C}(\mathbf{Sh}(\mathbb{R}^{n+1}))$ be a complex of sheaves such that $H^iF = 0$ for $i \notin [0,d]$, for some d. Then $H^iRp_!F = 0$ for $i \notin [0,d+1]$. In particular, for any $F \in \mathbf{Sh}(\mathbb{R}^n)$ we have $H^i_c(\mathbb{R}^n; F) \simeq 0$ if $i > n$.*

Proof. Using the truncation functors we can really assume that $F^i = 0$ for $i \notin [0,d]$. Then the complex G found in the example is p-soft resolution of F of length $d+1$ and the conclusion follows. ☐

11.2. **Calculus of fractions.**

Definition 11.18. A family $\mathcal{S}$ of morphisms in $\mathcal{A}$ is a left multiplicative system if

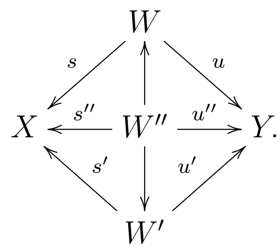
(i) any isomorphism belongs to $\mathcal{S}$,
(ii) if $f, g \in \mathcal{S}$ and $g \circ f$ is defined, then $g \circ f \in \mathcal{S}$,
(iii) for given morphisms f, s, $s \in \mathcal{S}$, as in the following diagram, there exist g, t, $t \in \mathcal{S}$, making the diagram commutative

$$\begin{array}{ccc}
X & \overset{g}{\dashrightarrow} & Y \\
{\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\
X' & \xrightarrow{f} & Y',
\end{array}$$

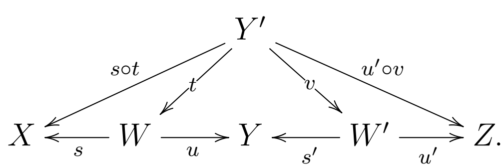
(iv) for two given morphisms $f, g\colon X \to Y$ in $\mathcal{A}$, if there exists $s \in \mathcal{S}$ such that $s \circ f = s \circ g$, then there exists $t \in \mathcal{S}$ such that $f \circ t = g \circ t$:

$$W \overset{t}{\dashrightarrow} X \xrightarrow{f,g} Y \xrightarrow{s} Z.$$

Proposition 11.19. *Let $\mathcal{A}$ be a category and $\mathcal{S}$ a left multiplicative system. Then $\mathcal{A}_{\mathcal{S}}$ can be described as follows. The set of objects is* $\mathrm{Ob}(\mathcal{A}_{\mathcal{S}}) = \mathrm{Ob}(\mathcal{A})$*. For $X, Y \in \mathrm{Ob}(\mathcal{A})$, we have* $\mathrm{Hom}_{\mathcal{A}_{\mathcal{S}}}(X,Y) = \{(W,s,u);\ s\colon W \to X$ *is in $\mathcal{S}$ and* $u\colon W \to Y$ *is in* $\mathcal{A}\}/\sim$*, where the equivalence relation $\sim$ is given by $(W,s,u) \sim (W',s',u')$ if there exists (W'',s'',u''), $s'' \in \mathcal{S}$, such that we have a commutative diagram*



The composition "$u's'^{-1}us^{-1}$" is visualized by the diagram



where t, v, $t \in \mathcal{S}$, are given by (iii) in Definition 11.18.

Let us go back to our abelian category $\mathcal{C}$.

Proposition 11.20. *Let Qis be the family of qis in $\mathbf{K}(\mathcal{C})$. Then Qis is a left (and right) multiplicative system.*

11.3. **Truncation functors.** Let $\mathcal{C}$ be an abelian category. For a given $n \in \mathbb{Z}$ we define $\tau_{\leq n}, \tau_{\geq n}\colon \mathbf{C}(\mathcal{C}) \to \mathbf{C}(\mathcal{C})$ by

$$\tau_{\leq n}(X) = \cdots \to X^{n-2} \to X^{n-1} \to \ker(d_X^n) \to 0 \to \cdots$$
$$\tau_{\geq n}(X) = \cdots \to 0 \to \operatorname{coker}(d_X^{n-1}) \to X^{n+1} \to X^{n+2} \to \cdots .$$

We have natural morphisms in $\mathbf{C}(\mathcal{C})$, for $n \leq m$,

$$\tau_{\leq n}(X) \to X, \qquad X \to \tau_{\geq n}(X),$$
$$\tau_{\leq n}(X) \to \tau_{\leq m}(X), \qquad \tau_{\geq n}(X) \to \tau_{\geq m}(X).$$

We have $H^i(\tau_{\leq n}(X)) \simeq H^i(X)$ for $i \leq n$ and $H^i(\tau_{\leq n}(X)) \simeq 0$ for $i > 0$. We have a similar result for $\tau_{\geq n}(X)$ and the above morphisms induce the tautological morphisms on the cohomology (that is, the identity morphism of H^i if both groups are non-zero, or the zero morphism).

In particular the functors $\tau_{\leq n}$, $\tau_{\geq n}$ send qis to qis and they induce functors, denoted in the same way, on $\mathbf{D}(\mathcal{C})$, together with the same morphisms of functors. We see from the definition, for any $X \in \mathbf{D}(\mathcal{C})$ and any $i \in \mathbb{Z}$:

$$\tau_{\leq i}\tau_{\geq i}(X) \simeq \tau_{\geq i}\tau_{\leq i}(X) \simeq H^i(X)[-i]. \tag{11.2}$$

Lemma 11.21. *Let $\mathcal{C}$ be an abelian category and let $X \in \mathbf{D}(\mathcal{C})$ be an objet concentrated in one degree i_0, that is, $H^i(X) \simeq 0$ if $i \neq i_0$. Then $X \simeq H^{i_0}(X)[-i_0]$.*

Proof. By the hypothesis and by the description of the cohomology of $\tau_{\leq n}(X)$, $\tau_{\geq n}(X)$, the morphisms $\tau_{\leq i_0}(X) \to X$ and $\tau_{\leq i_0}(X) \to \tau_{\geq i_0}(\tau_{\leq i_0}(X))$ are isomorphisms in $\mathbf{D}(\mathcal{C})$. Hence $X \simeq \tau_{\geq i_0}(\tau_{\leq i_0}(X))$ and we conclude with (11.2). □

11.4. **The case of cohomological dimension** 1**.** The next proposition describes $\mathbf{D}^-(\mathcal{C})$ when $\mathcal{C}$ has *cohomological dimension* 1, which means that $\mathrm{Ext}^i(X, Y) \simeq 0$ for all $i > 1$ and all $X, Y \in \mathcal{C}$. We first give some lemmas.

Lemma 11.22. *Let $\mathcal{C}$ be an abelian category. Then $Q \in \mathcal{C}$ is projective if and only if $\mathrm{Ext}^1(Q, M) \simeq 0$ for all $M \in \mathcal{C}$.*

Proof. The "only if" statement is a particular case of the fact $L^iF(Q) \simeq 0$ for $i > 0$ if Q is projective and F is a right exact functor.

Conversely, let $X \xrightarrow{p} Y \to 0$ be an epimorphism in $\mathcal{C}$. We set $M = \ker(p)$. Hence $0 \to M \to X \to Y \to 0$ is an exact sequence. The long cohomology exact sequence for the functor $\mathrm{Hom}(Q, \cdot)$ is written:

$$\cdots \to \mathrm{Hom}(Q, X) \to \mathrm{Hom}(Q, Y) \to \mathrm{Ext}^1(Q, M) \to \cdots$$

The hypothesis implies that $\mathrm{Hom}(Q, X) \to \mathrm{Hom}(Q, Y)$ is an epimorphism, which proves that Q is projective. □

Lemma 11.23. *Let $\mathcal{C}$ be an abelian category. We assume that for all $X, Y \in \mathcal{C}$ we have $\mathrm{Ext}^2(X, Y) \simeq 0$. Let P be a projective object and let $0 \to Q \xrightarrow{i} P$ be a monomorphism. Then Q is projective.*

Proof. We set $Z = \mathrm{coker}(i)$. Let $M \in \mathcal{C}$ be any object. As in the proof of Lemma 11.22 we have the long exact sequence

$$\cdots \mathrm{Ext}^1(P, M) \to \mathrm{Ext}^1(Q, M) \to \mathrm{Ext}^2(Z, M) \to \cdots$$

Since P is projective, the first term vanishes by Lemma 11.22. The second term vanishes by hypothesis. Hence $\mathrm{Ext}^1(Q, M) \simeq 0$ and Q is projective by Lemma 11.22. □

Exercise 11.24. Let $\mathcal{C}$ be an abelian category with enough projectives such that for all $X, Y \in \mathcal{C}$ we have $\mathrm{Ext}^2(X, Y) \simeq 0$. Prove that $\mathrm{Ext}^i(X, Y) \simeq 0$ for all $i \geq 2$ and all $X, Y \in \mathcal{C}$.

Exercise 11.25. Give a generalization of Lemma 11.23 and Exercise 11.24 with 2 replaced by any $k \geq 2$.

Lemma 11.26. *Let $\mathcal{C}$ be an abelian category and let $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ be an exact sequence in $\mathcal{C}$. We assume that p has a splitting, that is, $j\colon C \to B$ such that $p \circ j = \mathrm{id}_B$. Then i has a splitting, that is, $q\colon B \to A$ such that $q \circ i = \mathrm{id}_A$. Conversely, if i splits, then p splits.*

Proposition 11.27. *Let $\mathcal{C}$ be an abelian category. We assume that for all $X, Y \in \mathcal{C}$ we have $\mathrm{Ext}^2(X,Y) \simeq 0$. We also assume that $\mathcal{C}$ has enough projectives. Then for all $X \in \mathbf{D}^-(\mathcal{C})$ we have $X \simeq \bigoplus_{k\in\mathbb{Z}}(H^k(X))[-k]$. (We remark that $\bigoplus_{k\in\mathbb{Z}}(H^k(X))[-k]$ is the complex L given by $L^k = H^k(X)$ and $d_L^k = 0$ for all $k \in \mathbb{Z}$.)*

Proof. (i) We use the notation L of the proposition. Since $\mathcal{C}$ has enough projectives we can find $P \in \mathbf{D}^-(\mathcal{C})$ such that P^k is projective for all $k \in \mathbb{Z}$ and an isomorphism $X \simeq P$ in $\mathbf{D}^-(\mathcal{C})$.

We will define a morphism $u\colon P \to L$ in $\mathbf{C}(\mathcal{C})$ such that u is a qis. Then u induces the required isomorphism in $\mathbf{D}(\mathcal{C})$. This is the same as giving, for each i, a morphism $u^i\colon P^i \to L^i$ such that $d_P^{i-1} \circ u^i = 0$ and the induced morphism $Z^i(P)/B^i(P) \to L^i$ is an isomorphism.

(ii) We recall that we have monomorphisms $0 \to Z^i(P) \to P^i$ and $0 \to B^i(P) \to Z^i(P)$. By the hypothesis on $\mathcal{C}$ and by Lemma 11.23 we deduce that $Z^i(P)$ and then $B^i(P)$ are projective. By (3.1) we have the exact sequence

$$0 \to Z^i(P) \xrightarrow{a^i} P^i \xrightarrow{b^i} B^{i+1}(P) \to 0. \tag{11.3}$$

Since $B^{i+1}(P)$ is projective, the morphism b^i in (11.3) has a splitting and, by Lemma 11.26, there exists $\alpha^i\colon P^i \to Z^i(P)$ such that $\alpha^i \circ a^i = \mathrm{id}$.

Let $q^i\colon Z^i(P) \to H^i(P) = L^i$ be the natural morphism. We define $u^i\colon P^i \to L^i$ as $u^i = q^i \circ \alpha^i$. Since d^{i-1} factorizes as

$$P^{i-1} \xrightarrow{f^{i-1}} B^i(P) \xrightarrow{g^i} Z^i(P) \xrightarrow{a^i} P^i$$

we have $u^i \circ d^{i-1} = q^i \circ g^i \circ f^{i-1}$ and this vanishes because $q^i \circ g^i = 0$. We see also that the morphism $Z^i(P)/B^i(P) \to L^i$ induced by u^i is the identity morphism of $H^i(P)$. This concludes the proof. ☐

Example 11.28. We have seen that **Ab** has enough injectives and that an abelian group is injective if and only if it is divisible. It follows easily that a quotient of an injective abelian group is again injective. We deduce that any abelian group M has an injective resolution of length 1: $0 \to M \to I^0 \to I^1 \to 0$. Hence $\mathrm{Ext}^2(N, M) \simeq 0$ for all $M, N \in \mathbf{Ab}$.

11.5. **Example of sheaf computation.** Let X be a Hausdorff and locally compact space, let $Z \subset X$ be a closed subset and $U = X \setminus Z$. We let $j\colon U \to X$, $i\colon Z \to X$ be the inclusions. For $F \in \mathbf{Sh}(X)$ we set

$$F_Z = i_! i^{-1}(F), \qquad F_U = j_! j^{-1}(F).$$

We apply Proposition 9.11 with $f = i$ or $f = j$. Then $f^{-1}(y)$ is empty or a point and $\Gamma_c(f^{-1}(y); F|_{f^{-1}(y)})$ is 0 or F_y. This gives

$$(F_Z)_x = \begin{cases} F_x & \text{if } x \in Z, \\ 0 & \text{if } x \notin Z, \end{cases} \qquad (F_U)_x = \begin{cases} F_x & \text{if } x \in U, \\ 0 & \text{if } x \notin U, \end{cases} \tag{11.4}$$

We remark that i is proper, hence $i_! = i_*$ and $F_Z = i_* i^{-1}(F)$. By the adjunction (i^{-1}, i_*) we have a natural morphism $a\colon F \to F_Z$. By (11.4) we can see that a_x is the identity morphism for $x \in Z$ and $a_x = 0$ for $x \notin Z$. Hence a_x is always surjective and a is an epimorphism.

Lemma 11.29. *We have $F_U|_U \simeq F|_U$ and there exists a unique morphism $b\colon F_U \to F$ such that $b|_U\colon F_U|_U \to F_U$ is the identity morphism.*

Proof. If $V \subset U$ we can see on the definition that $F_U(V) = F(V)$, which proves the first assertion.

Now we pick any open subset $V \subset X$ and $s \in F_U(V)$. By definition $F_U(V) \subset (j_* j^{-1} F)(V) = F(U \cap V)$. The inclusion map $\operatorname{supp}(s) \to V$ is proper. Hence $W = V \setminus \operatorname{supp}(s)$ is open: indeed, for $x \in W$ we choose a compact neighborhood $C \subset V$ of x; then $C \cap \operatorname{supp}(s)$ is compact and $C \setminus (C \cap \operatorname{supp}(s))$ is open and contains x; hence W contains an open neighborhood of any of its point. Of course $s|_{U \cap V \cap W} = 0$. Since F is a sheaf, there exists a unique $\tilde{s} \in F(V)$ such that $\tilde{s}|_{U \cap V} = s|_{U \cap V}$ and $\tilde{s}|_W = 0$.

We define $b(V)\colon F_U(V) \to F(V)$ by $b(V)(s) = \tilde{s}$. When V runs over the open subsets, we can see that gives a sheaf morphism. □

Using (11.4) we see that the following *excision* sequence is exact

$$0 \to F_U \to F \to F_Z \to 0. \tag{11.5}$$

Lemma 11.30. *Let S^n be the sphere of dimension n. Then*

$$H^i(S^n; \mathbf{k}_{S^n}) \simeq \begin{cases} \mathbf{k} & \text{for } i = 0, n, \\ 0 & \text{else.} \end{cases}$$

Proof. We choose a point $x \in S^n$ and set $Z = \{x\}$, $U = S^n \setminus Z$. Let $i\colon Z \to S^n$ $j\colon U \to S^n$ be the inclusions and $a\colon S^n \to \{pt\}$ be the map to the point. We have $a_* = a_!$ since S^n is compact. Then $\Gamma(S^n; F_U) = a_! j_! (j^{-1}(F)) = \Gamma_c(U; j^{-1}(F))$ and $\Gamma(S^n; F_Z) = a_* i_* (i^{-1}(F)) = \Gamma(Z; i^{-1}(F)) = F_x$.

By Proposition 9.11 with $f = i$ or $f = j$ and the fact that $f^{-1}(y)$ is either epmpty or a point, we see that the functors $i_!$ and $j_!$ are exact. Hence $Ri_! = i_!$, $Rj_! = j_!$. We see also that they send soft sheaves to soft sheaves. Hence $R(a \circ i)_! \simeq Ra_! \circ Ri_!$ and $R(a \circ j)_! \simeq Ra_! \circ Rj_!$. It follows that $\mathrm{R}\Gamma(S^n; F_U) \simeq \mathrm{R}\Gamma_c(U; F|_U)$ and $\mathrm{R}\Gamma(S^n; F_Z) \simeq \mathrm{R}\Gamma(Z; F|_Z)$.

For $F = \mathbf{k}_{S^n}$, the sequence (11.5) becomes $0 \to \mathbf{k}_U \to \mathbf{k}_{S^n} \to \mathbf{k}_Z \to 0$. We deduce the long cohomology sequence $\cdots \to H^i_c(U; \mathbf{k}_U) \to H^i(S^n; \mathbf{k}_{S^n}) \to H^i(Z; \mathbf{k}_Z) \to \cdots$ Since Z is a point we conclude with Proposition 11.13. □

12. Exercices

Exercice 12.1. On a vu la définition de la catégorie opposée dans la Définition 4.1. Montrer que pour **Set** la catégorie des ensembles, **Set** et $\mathbf{Set}^{\mathrm{op}}$ ne sont pas équivalentes. (On peut sintéresser entre autres aux objets initiaux/terminuax).

Exercice 12.2. Soit $\mathbf{Ab}_f$ la catégorie des groupes abéliens finis. On rappelle que tout $G \in \mathbf{Ab}_f$ a une décomposition $G \simeq \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z} \times \cdots \times \mathbb{Z}/n_k\mathbb{Z}$. On note $G^* = \mathrm{Hom}_{\mathbf{Ab}}(G, S^1)$ où S^1 est le groupe (abélien) des nombres complexes de module 1. On peut voir qu'il existe un isomorphisme (non canonique) $(\mathbb{Z}/n\mathbb{Z})^* \simeq \mathbb{Z}/n\mathbb{Z}$, puis $G^* \simeq G$ pour tout $G \in \mathbf{Ab}_f$. Montrer que le morphisme naturel $G \to (G^*)^*$, $g \mapsto (\phi \mapsto \phi(g))$, est un isomorphisme. En déduire $\mathbf{Ab}_f^{\mathrm{op}} \simeq \mathbf{Ab}_f$.

Exercice 12.3. Montrer que **Ab** et $\mathbf{Ab}^{\mathrm{op}}$ ne sont pas équivalentes.

Exercice 12.4. Soit $\mathbf{Ab}_f$ la catégorie des groupes abéliens finis. Montrer que $\mathbf{Ab}_f$ n'a aucun objet projectif.

Exercice 12.5. On fixe $\alpha \in \mathbb{C}$, non nul. Soit U un ouvert de $\mathbb{C}^*$. On définit $F_\alpha(U)$ comme l'ensemble des fonctions $h\colon U \to \mathbb{C}$ telles que chaque $z_0 \in U$ a un voisinage $V \subset U$ sur lequel on a $h(z) = \lambda \exp(\alpha \log(z))$ où $\lambda \in \mathbb{C}$ et $\log(z)$ est une branche du logarithme définie sur V. Autrement dit, localement $f(z) = \lambda z^\alpha$. (Notons que si $\alpha \in \mathbb{Z}$ on peut toujours définir z^α alors qu'on n'a pas forcément une branche du log définie sur V tout entier.)

Vérifier que $F_\alpha(U)$ est un espace vectoriel. Montrer que, si U est connexe, alors $\dim F_\alpha(U)$ est 0 ou 1.

Vérifer que $U \mapsto F_\alpha(U)$ est un faisceau sur $\mathbb{C}^*$.

Pour $\theta \in [0, 2\pi[$ on note W_θ le plan privé de la demi-droite $\mathbb{R}_{\geq 0} \cdot \exp(i\theta)$. Montrer que $F_\alpha|_{W_\theta}$ est isomorphe au faisceau constant $\mathbb{C}_{W_\theta}$.

Montrer que F_α n'est pas isomorphe au faisceau constant $\mathbb{C}_{\mathbb{C}^*}$ si $\alpha \notin \mathbb{Z}$.

Exercice 12.6. Soit F un faisceau sur un espace topologique X. Soit $u\colon \mathbb{Z}_X \to F$ un morphisme de faisceaux. En prenant les sections sur X on a $\Gamma(X; u)\colon \Gamma(X; \mathbb{Z}_X) \to \Gamma(X; F)$ et on pose $\phi(u) = \Gamma(X; u)(1) \in \Gamma(X; F)$ (où 1 désigne la fonction constante de valeur 1). Vérifier que $\phi\colon \mathrm{Hom}_{\mathbf{Sh}(X)}(\mathbb{Z}_X, F) \to \Gamma(X; F)$ est un isomorphisme (on peut utiliser l'adjonction (a^{-1}, a_*), $a\colon X \to \{\mathrm{pt}\}$).

Soit U un ouvert de X. On définit un préfaisceau $P\mathbb{Z}_U$ par $P\mathbb{Z}_U(V) = \mathbb{Z}$ si $V \subset U$ et $P\mathbb{Z}_U(V) = 0$ sinon. Soit $\mathbb{Z}_U$ le faisceau associé à $P\mathbb{Z}_U$. Montrer que $\mathrm{Hom}_{\mathbf{Psh}(X)}(P\mathbb{Z}_U, F) \simeq F(U)$ puis $\mathrm{Hom}_{\mathbf{Sh}(X)}(\mathbb{Z}_U, F) \simeq F(U)$.

Exercice 12.7. Soit X un espace topologique connexe. Soit U un ouvert connexe de X et A un groupe abélien. Vérifer que $A_U(U) = A$. Soit $U, V \subset X$ deux ouverts connexes tels que $U \cup V = X$. Soit F un faisceau sur X et A, B deux groupes abéliens tels qu'il existe des isomorphismes $F|_U \simeq A_U$, $F|_V \simeq B_V$. Montrer que $A \simeq B$. Soit $x \in U$. On a $F_x \simeq (A_U)_x = A$. Montrer que le morphisme naturel $r\colon F(X) \to F_x = A$ est injectif. On suppose que r est un isomorphisme. Montrer que $F \simeq A_X$ (on peut s'inspirer de l'exercice (12.6) pour construire un morphisme de A_X vers F: par adjonction $\mathrm{Hom}_{\mathbf{Sh}(X)}(A_X, F) \simeq \mathrm{Hom}_{\mathbf{Ab}}(A, F(X))$).

Exercice 12.8. Soit X un espace topologique localement compact (si on préfère on peut supposer que X est un fermé de $\mathbb{R}^n$).

Soit $j\colon W \hookrightarrow X$ l'inclusion d'un sous-ensemble de X. On met sur W la topologie induite. On note $\mathbb{Z}_W \in \mathbf{Sh}(W)$ le faisceau constant.

Montrer que, si W est ouvert ou fermé et $F \in \mathbf{Sh}(W)$, on a $(j_! F)_x = 0$ pour $x \notin W$ et $j^{-1}(F) = F$.

Dans $\mathbb{R}^2$ on définit $W = \{0\} \cup]0,1[^2$. Montrer que $(j_!\mathbb{Z}_W)_0 = 0$.

On dit que W est localement fermé si on peut écrire $W = U \cap Z$ avec U ouvert et Z fermé (c'est équivalent à, pour chaque $x \in W$, il existe Ω ouvert de $\mathbb{R}^n$ tel que $W \cap \Omega$ est fermé dans Ω). En écrivant $j = j_2 \circ j_1$, $j_1\colon W \to U$, $j_2\colon U \to \mathbb{R}^n$ vérifier qu'on a encore, pour $F \in \mathbf{Sh}(W)$, $(j_! F)_x = 0$ pour $x \notin W$ et $j^{-1}(F) = F$.

Notation 12.9. Soit X un espace topologique localement compact (si on préfère on peut supposer que X est un fermé de $\mathbb{R}^n$). Soit $U \subset X$ un ouvert et $Z = X \setminus U$. On pose $\mathbb{Z}_{X,U} = j_!(\mathbb{Z}_U)$ où $j\colon U \to X$ est l'inclusion et de même $\mathbb{Z}_{X,Z} = i_!(\mathbb{Z}_Z)$ où i est l'inclusion de Z. Ainsi $\mathbb{Z}_{X,U} = (\mathbb{Z}_X)_U$, $\mathbb{Z}_{X,Z} = (\mathbb{Z}_X)_Z$ avec les notations de §11.5. On a la suite exacte (11.5) $0 \to \mathbb{Z}_{X,U} \to \mathbb{Z}_X \to \mathbb{Z}_{X,Z} \to 0$. (Il n'est pas trop dur de vérifier qu'on a bien les mêmes faisceaux que dans les exercices 2.32 et 2.40.)

Exercice 12.10. Soit C le cercle de centre $(0,0)$ et de rayon 1 dans $\mathbb{R}^2$ et $p\colon \mathbb{R}^2 \to \mathbb{R}$ la projection $(x,y) \mapsto x$. On utilise les notations 12.9.

Soit $I = [-1,1]$, $J =]-1,1[$. On veut montrer que $p_*(\mathbb{Z}_{\mathbb{R}^2,C}) = \mathbb{Z}_{\mathbb{R},I} \oplus \mathbb{Z}_{\mathbb{R},J}$.

(i) Soit D le disque fermé de centre $(0,0)$ et de rayon 1. Soit $I' \subset C$ le demi- cercle $I' = C \cap (\mathbb{R} \times [0,+\infty[)$. On remarque que I' est fermé dans C et C fermé dans D, d'où des morphismes $u\colon \mathbb{Z}_{\mathbb{R}^2,D} \to \mathbb{Z}_{\mathbb{R}^2,C}$ et $v\colon \mathbb{Z}_{\mathbb{R}^2,C} \to \mathbb{Z}_{\mathbb{R}^2,I'}$.

Montrer que $p_*(\mathbb{Z}_{\mathbb{R}^2,D}) = \mathbb{Z}_{\mathbb{R},I}$ et $p_*(\mathbb{Z}_{\mathbb{R}^2,I'}) = \mathbb{Z}_{\mathbb{R},I}$. Montrer que $p_*(v \circ u) = \mathrm{id}_{\mathbb{Z}_{\mathbb{R},I}}$.

Soit $J' = C \setminus I'$. Montrer que $p_*(\mathbb{Z}_{\mathbb{R}^2,J'}) = \mathbb{Z}_{\mathbb{R},J}$.

(ii) On a la suite exacte $0 \to \mathbb{Z}_{\mathbb{R}^2,J'} \to \mathbb{Z}_{\mathbb{R}^2,C} \to \mathbb{Z}_{\mathbb{R}^2,I'} \to 0$. Montrer qu'on a aussi la suite exacte $0 \to \mathbb{Z}_{\mathbb{R},J} \to p_*(\mathbb{Z}_{\mathbb{R}^2,C}) \to \mathbb{Z}_{\mathbb{R},I} \to 0$.

(iii) Montrer le fait général: dans une catégorie abélienne, si on a une suite exacte $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ et un morphisme $q\colon B \to A$ tel que $q \circ i = \mathrm{id}_A$ (on dit que la suite est scindée), alors le morphisme $\binom{q}{p}\colon B \to A \oplus C$ est un isomorphisme. (On peut montrer que les noyaux et conoyaux de $\binom{q}{p}$ sont nuls: pour $f\colon X \to B$ tel que $\binom{q}{p} \circ f = 0$ on montre d'abord que f factorise par $i\colon A \to B$, pour $(a,c)\colon A \oplus C \to Y$ tel que $(a,c) \circ \binom{q}{p} = 0$, on voit d'abord que $a = (a,c) \circ \binom{q}{p} \circ i = 0$.)

Exercice 12.11. Soit $F\colon \mathcal{C} \to \mathcal{C}'$ un foncteur *exact* entre catégories abéliennes. On suppose que $\mathcal{C}$ a assez d'injectifs. Montrer que $R^iF(X) = 0$ pour tout $i > 0$ et tout $X \in \mathcal{C}$. (On peut montrer que F envoie une suite exacte (longue) sur une suite exacte ou utiliser le Lemme 10.6.)

Exercice 12.12. Soit $f\colon X \to Y$ une application continue entre espaces topologiques localement compacts. On suppose que, pour tout $y \in Y$, $f^{-1}(y)$ est un ensemble fini. Montrer que $f_!\colon \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ est un foncteur exact.

Exercice 12.13. Soit $X = S^{2n-1}$ la sphère de dimension $2n-1$. On considère un groupe cyclique $G = \mathbb{Z}/k\mathbb{Z}$ agissant librement sur X (voir exemple ci-dessous) et on note $Y = X/G$ le quotient, $q\colon X \to Y$ l'application quotient. On veut calculer $H^*(Y;\mathbb{Z}_Y)$.

(1) On pose $F = q_*(\mathbb{Z}_X)$. En utilisant les exercices 12.11 et 12.12 montrer que $H^i(Y;F) \simeq H^i(X;\mathbb{Z}_X)$.

(2) Soit $V \subset Y$ un ouvert et $U = q^{-1}(V)$. Le groupe G préserve U; on note $g \cdot x$ l'action, pour $g \in G$, $x \in U$. Soit $\mathcal{F}(U)$ l'ensemble des fonctions sur U; G a une action induite sur $\mathcal{F}(U)$ définie par $g \cdot f(x) = f(g^{-1} \cdot x)$, où $f \in \mathcal{F}(U)$, $g \in G$, $x \in U$ (pourquoi g^{-1} à l'intérieur de f?). Déduire que G agit sur $F(V)$.

Soit $y \in V$. On a $F_y \simeq \bigoplus_{x \in q^{-1}(y)} (\mathbb{Z}_X)_x \simeq \mathbb{Z}^{q^{-1}(y)}$. Le groupe G agit sur $q^{-1}(y)$ et donc sur $F_y \simeq \mathbb{Z}^{q^{-1}(y)}$. Montrer que nos actions commutent avec l'application naturelle $F(V) \to F_y$.

(3) On remarque que $\mathbb{Z}_X \simeq q^{-1}(\mathbb{Z}_Y)$. En utilisant l'adjonction (q^{-1}, q_*) on a un morphisme $a\colon \mathbb{Z}_Y \to F$. Pour $y \in Y$ on a les isomorphismes naturels $(\mathbb{Z}_Y)_y \simeq \mathbb{Z}$ et $F_y \simeq \mathbb{Z}^{q^{-1}(y)}$. Décrire a_y via ces isomorphismes.

(4) Soit $y_i \in Y$ et V_i un voisinage connexe de y_i assez petit pour que $q^{-1}(V_i) = \bigsqcup_{j=1}^k U_i^k$ (union disjointe de k composantes connexes;

c'est possible car q est un revêtement à $|G| = k$ feuillets). On a $F|_{V_i} \simeq \mathbb{Z}^k_{V_i}$ et on définit $b_i\colon F|_{V_i} \to \mathbb{Z}_{V_i}$ par $b_i(V)(f_1, \dots, f_k) = f_1 + \cdots + f_k$, pour $V \subset V_i$. On recouvre Y par de tels ouverts V_i, $i \in I$. Vérifier que les b_i se recollent en un morphisme de faisceaux $b\colon F \to \mathbb{Z}_Y$ (on peut utiliser l'exercice 7.7). Montrer que $b \circ a\colon \mathbb{Z}_Y \to \mathbb{Z}_Y$ est la multiplication par k.

(5) On a vu que G agit sur $F(V)$, pour tout $V \subset Y$. Pour $g \in G$ on note $\mu_g(V)\colon F(V) \to F(V)$ l'action de g. Vérifier que les $\mu_g(V)$ donnent un morphisme de faisceau $\mu_g\colon F \to F$. On note $\mathbf{1}$ un générateur de G. On définit $u\colon F \to F$, $u = \mathrm{id}_F - \mu_{\mathbf{1}}$. Montrer qu'on a la suite exacte $0 \to \mathbb{Z}_Y \xrightarrow{a} F \xrightarrow{u} F \xrightarrow{b} \mathbb{Z}_Y \to 0$.

(6) On note $L = \mathrm{coker}(a) = \ker(b)$ et on a deux suites exactes courtes $0 \to \mathbb{Z}_Y \xrightarrow{a} F \to L \to 0$, $0 \to L \to F \xrightarrow{b} \mathbb{Z}_Y \to 0$. On connaît $H^i(Y;F) = H^i(X;\mathbb{Z}_X)$. Vérifier que $H^0(Y;a) = \mathrm{id}_Z$ et $H^0(Y;b) = k\,\mathrm{id}_Z$. Montrer par récurrence

$$\begin{aligned} H^0(Y;\mathbb{Z}_Y) &= \mathbb{Z}, \\ H^i(Y;\mathbb{Z}_Y) &= \mathbb{Z}/k\mathbb{Z}, \quad \text{pour } i \text{ pair et } 0 < i < 2n-2, \\ H^{2n-1}(Y;\mathbb{Z}_Y) &= \mathbb{Z}, \\ H^i(Y;\mathbb{Z}_Y) &= 0, \quad \text{sinon} \end{aligned}$$

(on sait que $H^i(Y;F') = 0$ pour tout faisceau F' et $i > \dim Y = 2n-1$).

Les exemples de variétés Y comme dans l'exercice sont les espaces lenticulaires. On voit la sphère $X = S^{2n-1}$ dans $\mathbb{C}^n = \mathbb{R}^{2n}$. On note S^1 le cercle unité de $\mathbb{C}$. Alors $(S^1)^n$ agit sur $\mathbb{C}^n$ par multiplication $(s_1, \dots, s_n) \cdot (z_1, \dots, z_n) = (s_1 z_1, \dots, s_n z_n)$ et cette action préserve X. On définit une injection $i\colon \mathbb{Z}/k\mathbb{Z} \to (S^1)^n$ de façon que la projection sur chaque coordonnée soit encore injective: on choisit $p_1, \dots, p_n$ des entiers tous premiers à k et on définit $i([m]) = (\zeta^{mp_1}, \dots, \zeta^{mp_n})$, où $\zeta = \exp(2i\pi/k)$. Alors l'action de $\mathbb{Z}/k\mathbb{Z}$ sur X est libre (seul 0 a des points fixes). On note $L_{k;p_1,\dots,p_n} = X/(\mathbb{Z}/k\mathbb{Z})$. Plus spécialement en dimension 3 on note $L_{p/k} = L_{k;1,p}$. Ce sont les premiers exemples de variétés différentielles homotopes mais non homéomorphes (par exemple $L_{1/7}$ et $L_{2/7}$ sont homotopes mais non homéomorphes).

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13. Examen

Exercice 13.1. Soit $\mathcal{C}$ une catégorie abélienne et soit $X \xrightarrow{f} Y \xrightarrow{g} Z$ deux morphismes dans $\mathcal{C}$. On suppose que $\ker(g \circ f) = 0$. Montrer que $\ker(f) = 0$.

Exercice 13.2.
(1) Soit X un espace topologique, F un faisceau sur X, $U \subset X$ un ouvert et $s \in F(U)$. Soit $x \in U$ tel que $s_x = 0$. Montrer qu'il existe un voisinage $V \subset U$ de x tel que $s|_V = 0$.

(2) Donner un exemple de faisceau F sur $X = \mathbb{R}$ et tel que $F_0 = 0$ mais 0 n'a aucun voisinage ouvert V tel que $F|_V = 0$.

Exercice 13.3. Soit $\mathcal{C}, \mathcal{C}'$ des categories abéliennes et soit $R \colon \mathcal{C}' \to \mathcal{C}$, $L \colon \mathcal{C} \to \mathcal{C}'$ des foncteurs additifs tels que R est adjoint à droite de L. Montrer que R est exact à gauche.

Exercice 13.4.
(1) Soit Z un ensemble muni de la topologie discrète (chaque point forme un ensemble ouvert). Soit F un faisceau sur Z. Vérifier que $\Gamma_c(Z; F) \simeq \bigoplus_{z \in Z} F_z$.

(2) Soit $f \colon X \to Y$ une application continue entre espaces topologiques localement compacts. On suppose que, pour tout $y \in Y$, $f^{-1}(y)$ est un ensemble discret. Montrer que $f_! \colon \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ est un foncteur exact.

Exercice 13.5. Dans $\mathbb{R}^2$ on définit $W = \{0\} \cup]0,1[^2$. Montrer que $(j_! \mathbb{Z}_W)_0 = 0$. Est-ce que $j_!$ est exact?

Exercice 13.6. Soit $\mathbb{Z}_\mathbb{R}, \mathbb{Z}_{\mathbb{R},[0,1]} \in \mathbf{Sh}(\mathbb{R})$ respectivement le faisceau constant sur $\mathbb{R}$ et le faisceau constant sur l'intervalle $[0,1]$. Montrer que $\mathrm{Hom}(\mathbb{Z}_{\mathbb{R},[0,1]}, \mathbb{Z}_\mathbb{R}) = 0$.

Exercice 13.7. Soit $\mathcal{C}$ une catégorie abélienne et soit $\mathrm{Mor}(\mathcal{C})$ la catégorie des morphismes de $\mathcal{C}$. On rappelle que les objects de $\mathrm{Mor}(\mathcal{C})$ sont les morphismes $(X \xrightarrow{f} X')$ de $\mathcal{C}$ et les morphismes de $\mathrm{Mor}(\mathcal{C})$ sont les carrés commutatifs: $\mathrm{Hom}_{\mathrm{Mor}(\mathcal{C})}((X \xrightarrow{u} X'), (Y \xrightarrow{v} Y')) = \{(f, f');\ f \colon X \to Y,\ f' \colon X' \to Y',\ v \circ f = f' \circ u\}$.

On admet que $\mathrm{Mor}(\mathcal{C})$ est abélienne et $\ker(f, f') = (\ker(f) \to \ker(f'))$, $\mathrm{coker}(f, f') = (\mathrm{coker}(f) \to \mathrm{coker}(f'))$.

(1) Montrer que $\ker \colon \mathrm{Mor}(\mathcal{C}) \to \mathcal{C}$, $(X \xrightarrow{u} X') \mapsto \ker(u)$ est un foncteur exact à gauche.

(2) Soit $\mathcal{E} \subset \mathrm{Mor}(\mathcal{C})$ la famille des épimorphismes. Montrer que $\mathcal{E}$ est ker-injective.

On rappelle qu'une famille $\mathcal{J}$ dans une catégorie abélienne $\mathcal{D}$ est F-injective si

(i) pour tout $X \in \mathrm{Ob}(\mathcal{D})$ il existe $J \in \mathcal{J}$ et un monomorphisme $0 \to X \to J$,

(ii) pour toute suite exacte $0 \to X' \to X \to X'' \to 0$ dans $\mathcal{D}$, si $X' \in \mathcal{J}$ et $X \in \mathcal{J}$, alors $X'' \in \mathcal{J}$,

(iii) pour toute suite exacte $0 \to X' \to X \to X'' \to 0$ dans $\mathcal{D}$, avec $X', X, X'' \in \mathcal{J}$, la suite $0 \to F(X') \to F(X) \to F(X'') \to 0$ est exacte.

(3) Donner pour chaque $(X \xrightarrow{u} X') \in \mathrm{Mor}(\mathcal{C})$ une résolution à deux termes dans $\mathcal{E}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & X^0 & \longrightarrow & X^1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u^0} & & \downarrow{\scriptstyle u^1} & & \\ 0 & \longrightarrow & X' & \longrightarrow & X'^0 & \longrightarrow & X'^1 & \longrightarrow & 0 \end{array}$$

et montrer que $R^1 \ker = \mathrm{coker}$.