SHORT LECTURE ON SHEAVES AND DERIVED CATEGORIES

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1. Sheaves

In this section $\mathbf{k}$ is a given ring.

1.1. **Definition.** A presheaf P (of $\mathbf{k}$-modules) on a topological space X is the data of $\mathbf{k}$-modules $P(U)$ for all open subsets U of X together with linear maps $r_U^V \colon P(U) \to P(V)$ for all inclusions $V \subset U$ such that $r_V^W \circ r_U^V = r_U^W$ for $W \subset V \subset U$. For a *section* $s \in P(U)$ we usually set $s|_V = r_U^V(s)$. A sheaf F is a presheaf such that, for any covering $U = \bigcup_{i\in I} U_i$ and sections $s_i \in F(U_i)$ satisfying $s_i|_{U_{ij}} = s_i|_{U_{ij}}$, there exists a unique $s \in F(U)$ such that $s_i = s|_{U_i}$.

We often set $\Gamma(U; F) = F(U)$.

The stalk of a presheaf at $x \in X$ is $P_x = \varinjlim_{U \ni X} P(U)$, where U runs over the open neighborhoods of x.

A morphism of presheaves $f \colon P \to P'$ is the data of groups morphisms $f(U) \colon P(U) \to P'(U)$ which commute with the restriction maps, that is, $r'_{V,U} \circ f(U) = f(V) \circ r_{V,U}$, for all $V \subset U \subset X$. A morphism of sheaves is a morphism of the underlying presheaves.

We denote by $\mathrm{Mod}(\mathbf{k}_X)$ the category of sheaves of $\mathbf{k}$-modules on X.

Examples 1.1. (i) The constant sheaf $\mathbf{k}_X$ on X is defined by $\mathbf{k}_X(U) = \{f \colon U \to \mathbf{k};\ f \text{ is locally constant}\}$. If N is a $\mathbf{k}$-module, we define N_X, the constant sheaf with stalks N, in the same way.

(ii) If $Z \subset X$ is a closed subset, we define $\mathbf{k}_{X,Z}$ (or $\mathbf{k}_Z$ if X is understand) by $\mathbf{k}_{X,Z}(U) = \{f \colon U \cap Z \to \mathbf{k};\ f \text{ is locally constant}\}$.

A morphism u in $\mathrm{Mod}(\mathbf{k}_X)$ is an isomorphism if and only if u_x is an isomorphism for all $x \in X$.

Lemma 1.2 (Associated sheaf of a presheaf)**.** *Let X be a topological space and let $P \in \mathcal{P}(X)$. There exist a sheaf P^a and a morphism of* *presheaves* $u\colon P \to P^a$ *such that* u_x *is an isomorphism, for each* $x \in X$*. Moreover the pair* (P^a, u) *is unique up to isomorphism.*

Any morphism v in $\mathrm{Mod}(\mathbf{k}_X)$ has a kernel, given by $U \mapsto \ker v(U)$, and a cokernel, given by $(U \mapsto \operatorname{coker} v(U))^a$. The category $\mathrm{Mod}(\mathbf{k}_X)$ is abelian (which means that it is additive, kernel and cokernel exist and are well-behaved in the sense "$\ker(\operatorname{coker}(v)) \simeq \operatorname{coker}(\ker(v))$"). We can also check: a sequence $F \xrightarrow{u} G \xrightarrow{v} H$ in $\mathrm{Mod}(\mathbf{k}_X)$ is exact if and only if the sequences of stalks $F_x \xrightarrow{u_x} G_x \xrightarrow{v_x} H_x$ are exact for all $x \in X$.

1.2. **Operations.**

Proposition 1.3. *Let* F_i*,* $i \in I$*, be a family of sheaves in* $\mathrm{Mod}(\mathbf{k}_X)$*. Then the product* $\prod_{i\in I} F_i$ *and the sum* $\bigoplus_{i\in I} F_i$ *exist in* $\mathrm{Mod}(\mathbf{k}_X)$*. The product is the sheaf defined by* $\Gamma(U; \prod_{i\in I} F_i) = \prod_{i\in I} \Gamma(U; F_i)$ *for any open subset* U*. The sum is the sheaf associated with the presheaf* $U \mapsto \bigoplus_{i\in I} \Gamma(U; F_i)$*. For any* $x \in X$ *we have a canonical isomorphism*

$$(\bigoplus_{i\in I} F_i)_x \simeq \bigoplus_{i\in I} (F_i)_x. \tag{1.1}$$

Definition 1.4. For $F, G \in \mathrm{Mod}(\mathbf{k}_X)$ we define a sheaf $\mathcal{H}om(F, G) \in \mathrm{Mod}(\mathbf{k}_X)$, the *internal hom* sheaf, by

$$\Gamma(U; \mathcal{H}om(F, G)) = \mathrm{Hom}_{\mathrm{Mod}(\mathbf{k}_U)}(F|_U, G|_U).$$

We define the tensor product $F \otimes_{\mathbf{k}_X} G$ as the sheaf associated with the presheaf $U \mapsto F(U) \otimes_{\mathbf{k}} G(U)$.

We can prove

$$(F \otimes_{\mathbf{k}_X} G)_x \simeq F_x \otimes_{\mathbf{k}} G_x, \qquad \text{for all } x \in X. \tag{1.2}$$

Lemma 1.5. *The functor* $\mathcal{H}om(\cdot, \cdot)$ *is left exact in both arguments. The functor* $\cdot \otimes_{\mathbf{k}_X} \cdot$ *is right exact in both arguments, and exact if* $\mathbf{k}$ *is a field.*

Let $f\colon X \to Y$ be a continuous map between topological spaces.

Definition 1.6. For $F \in \mathrm{Mod}(\mathbf{k}_X)$ we define a sheaf $f_*F \in \mathrm{Mod}(\mathbf{k}_Y)$ by $(f_*F)(V) = F(f^{-1}(V))$ for any open subset $V \subset Y$, with the restriction maps naturally given by those of F (it is clear that f_*F is a presheaf and it is easy to check that it is actually a sheaf).

If $u\colon F \to G$ is a morphism in $\mathrm{Mod}(\mathbf{k}_X)$, we define $f_*u\colon f_*F \to f_*G$ by $(f_*u)(V) = u(f^{-1}(V))$. We obtain a functor $f_*\colon \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{Mod}(\mathbf{k}_Y)$.

Lemma 1.7. *For any continuous map* $f\colon X \to Y$*, the functor* f_* *is left exact.*

Definition 1.8. For $G \in \mathrm{Mod}(\mathbf{k}_Y)$ we define a presheaf $f^\dagger G$ on X by $(f^\dagger G)(U) = \varinjlim_{V \supset f(U)} G(V)$, where V runs over the open neighborhoods of $f(U)$ in Y. The restriction maps are naturally induced by those of G. We set $f^{-1}G = (f^\dagger G)^a$.

A morphism $u\colon F \to G$ induces morphisms on the inductive limits, $(f^\dagger u)(U)\colon (f^\dagger F)(U) \to (f^\dagger G)(U)$, for all $U \in \mathrm{Op}(X)$, which are compatible and define $f^\dagger u\colon f^\dagger F \to f^\dagger G$. We set $f^{-1}u = (f^\dagger u)^a$. We thus obtain a functor $f^{-1}\colon \mathrm{Mod}(\mathbf{k}_Y) \to \mathrm{Mod}(\mathbf{k}_X)$.

Lemma 1.9. *The functor f^{-1} is left adjoint to f_*. In particular there exist natural isomorphisms* $\mathrm{Hom}(f^{-1}G, F) \simeq \mathrm{Hom}(G, f_*F)$ *for all* $F \in \mathrm{Mod}(\mathbf{k}_X)$*,* $G \in \mathrm{Mod}(\mathbf{k}_Y)$*.*

When $f\colon X \to Y$ is an embedding we often write

$$G|_X := f^{-1}G.$$

If f is the inclusion of an open set, we have $(G|_X)(U) = G(U)$, for all $U \in \mathrm{Op}(X)$.

Example 1.10. Let X be a Hausdorff topological space and $Z \subset X$ a compact subset. Then, for any $F \in \mathrm{Mod}(\mathbf{k}_X)$ and $V \in \mathrm{Op}(Z)$, we have $(F|_Z)(V) \simeq \varinjlim_{U \supset V} F(U)$, where U runs over the open neighborhoods of V in X.

Lemma 1.11. *Let $f\colon X \to Y$ be a continuous map and let $x \in Y$. For any $F \in \mathrm{Mod}(\mathbf{k}_Y)$ we have a natural isomorphism $(f^{-1}F)_x \simeq F_{f(x)}$.*

Since the exactness of a sequence of sheaves can be checked in the stalks we deduce:

Lemma 1.12. *For any continuous map $f\colon X \to Y$, the functor f^{-1} is exact.*

1.3. **Locally closed subsets.** A subset W of X is locally closed subset if we can write $W = U \cap Z$ with U open and Z closed.

Lemma 1.13. *Let $W \subset X$ be a locally closed subset and $F \in \mathrm{Mod}(\mathbf{k}_X)$. Then there exists a unique sheaf $F_W \in \mathrm{Mod}(\mathbf{k}_X)$ such that $F_W|_W \simeq F|_W$ and $F_W|_{X\setminus W} \simeq 0$. Moreover we have $F_W \simeq F \otimes (\mathbf{k}_X)_W$.*

We set for short $\mathbf{k}_{X,W} = (\mathbf{k}_X)_W$ and even $\mathbf{k}_W = \mathbf{k}_{X,W}$ when it is clear that we consider sheaves on X.

Example 1.14. If W is closed in X, the sheaf $\mathbf{k}_W$ is already defined in Example 1.1. In general we have $\mathbf{k}_W(U) \simeq \{f\colon U \cap W \to \mathbf{k};\ f$ is locally constant and $\{x;\ f(x) \neq 0\}$ is closed in $U\}$.

Lemma 1.15 (Excision)**.** *Let $W \subset X$ be a locally closed subset and let $W' \subset W$ be a closed subset of W. Then W' and $W \setminus W'$ are locally closed in X and we have an exact sequence:*

$$0 \to \mathbf{k}_{W \setminus W'} \to \mathbf{k}_W \to \mathbf{k}_{W'} \to 0.$$

Lemma 1.16 (Mayer-Vietoris)**.** *Let $Z_1, Z_2 \subset X$ be closed subsets and $U_1, U_2 \subset X$ open subsets. We have exact sequences*

$$0 \to \mathbf{k}_{Z_1 \cup Z_2} \to \mathbf{k}_{Z_1} \oplus \mathbf{k}_{Z_2} \to \mathbf{k}_{Z_1 \cap Z_2} \to 0,$$
$$0 \to \mathbf{k}_{U_1 \cap U_2} \to \mathbf{k}_{U_1} \oplus \mathbf{k}_{U_2} \to \mathbf{k}_{U_1 \cup U_2} \to 0.$$

1.4. **Proper direct image.** A topological space X is locally compact if, for any $x \in X$ and any neighborhood U of x, there exists a compact neighborhood of x contained in U. Now we assume X, Y are Hausdorff and locally compact. Then a map $f\colon X \to Y$ is *proper* if the inverse image of any compact subset of Y is compact.

Definition 1.17. Let $f\colon X \to Y$ be a continuous map of Hausdorff and locally compact spaces. For $F \in \mathrm{Mod}(\mathbf{k}_X)$ we define a subsheaf $f_! F \in \mathrm{Mod}(\mathbf{k}_Y)$ of $f_* F$ by

$$(f_! F)(V) = \{s \in (f^{-1}(V));\ f|_{\mathrm{supp}\, s}\colon \mathrm{supp}(s) \to V \quad \text{is proper}\}$$

for any open subset $V \subset Y$. If $u\colon F \to G$ is a morphism in $\mathrm{Mod}(\mathbf{k}_X)$, the morphism $f_* u\colon f_* F \to f_* G$ sends $f_! F$ to $f_! G$. We obtain a functor $f_!\colon \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{Mod}(\mathbf{k}_Y)$.

If the map f itself is proper, then we have $f_! \xrightarrow{\sim} f_*$.

Lemma 1.18. *The functor $f_!$ is left exact.*

For $F \in \mathrm{Mod}(\mathbf{k}_X)$ and $U \in \mathrm{Op}(X)$ we set

$$\Gamma_c(U; F) = \{s \in F(V);\ \mathrm{supp}(s) \text{ is compact.}\}$$

We have $\Gamma_c(U; F) \simeq a_!(F|_U)$, where a is the projection $U \to \{\mathrm{pt}\}$.

Proposition 1.19. *Let $f\colon X \to Y$ be as in Definition 1.17. For any $F \in \mathrm{Mod}(\mathbf{k}_X)$ and $y \in Y$ we have*

$$(f_! F)_y \simeq \Gamma_c(f^{-1}(y); F|_{f^{-1}(y)}).$$

Example 1.20. In the situation of Lemma 1.13 let $j\colon Z \to X$ be the inclusion. Then $\mathbf{k}_{X,Z} \simeq j_! \mathbf{k}_Z$ and $F_Z \simeq j_! j^{-1} F$.

1.5. **Enough injectives in** $\mathrm{Mod}(\mathbf{k}_X)$**.** We first remark the following general result.

Lemma 1.21. *Let* $f\colon X \to Y$ *be a continuous map and assume that* $I \in \mathrm{Mod}(\mathbf{k}_X)$ *is injective. Then* $f_*I \in \mathrm{Mod}(\mathbf{k}_Y)$ *is injective.*

Proof. The injectivity of $f_*(I)$ means that the map

$$\mathrm{Hom}_{\mathcal{C}'}(G, f_*I) \to \mathrm{Hom}_{\mathcal{C}'}(F, f_*I)$$

is surjective, for all monomorphism $0 \to F \to G$ in $\mathrm{Mod}(\mathbf{k}_Y)$. Since f^{-1} is exact, $f^{-1}F \to f^{-1}G$ is also a monomorphism and the injectivity of I gives the surjectivity of

$$\mathrm{Hom}_{\mathcal{C}}(f^{-1}G, I) \to \mathrm{Hom}_{\mathcal{C}}(f^{-1}F, I).$$

The result follows since (f^{-1}, f_*) is an adjoint pair. □

Let X be a topological space and let X^d be the set X endowed with the discrete topology (that is, any subset is open). The identity map $i\colon X^d \to X$ is continuous. For any $F \in \mathrm{Mod}(\mathbf{k}_X)$ the adjunction (i^{-1}, i_*) gives a morphism

$$\varepsilon_F\colon F \to i_*i^{-1}F. \tag{1.3}$$

For $U \in \mathrm{Op}(X)$ we have $(i_*i^{-1}F)(U) \simeq \prod_{x\in U} F_x$. We deduce:

Lemma 1.22. *For any* $F \in \mathrm{Mod}(\mathbf{k}_X)$ *the adjunction morphism* (1.3) *is a monomorphism.*

We remark that sheaves on X^d are easy to describe: $\mathcal{P}_{\mathbf{k}}(X^d) \xrightarrow{\sim} \mathrm{Mod}(\mathbf{k}_{X^d}) \simeq (\mathrm{Mod}(\mathbf{k}))^{X^d}$, that is, a sheaf $F \in \mathrm{Mod}(\mathbf{k}_{X^d})$ is a family of $\mathbf{k}$-modules F_x indexed by X. The exactness of a sequence is checked pointwise. We deduce that, if F_x is injective in $\mathrm{Mod}(\mathbf{k})$ for all $x \in X$, then $F = \{F_x\}_{x\in X}$ is injective in $\mathrm{Mod}(\mathbf{k}_{X^d})$. In particular $\mathrm{Mod}(\mathbf{k}_{X^d})$ has enough injectives: for a given $F = \{F_x\}_{x\in X}$ we choose a monomorphism $F_x \to I_x$, for all $x \in X$, where I_x is injective (which is possible since $\mathrm{Mod}(\mathbf{k})$ has enough injectives). Then $I = \{I_x\}_{x\in X}$ is injective in $\mathrm{Mod}(\mathbf{k}_{X^d})$ and $F \to I$ is a monomorphism.

Proposition 1.23. *For any topological space* X*,* $\mathrm{Mod}(\mathbf{k}_X)$ *has enough injectives.*

Proof. Let $F \in \mathrm{Mod}(\mathbf{k}_X)$. We have remarked that $\mathrm{Mod}(\mathbf{k}_{X^d})$ has enough injectives. Hence there exists a monomorphism $i^{-1}F \to I$ in $\mathrm{Mod}(\mathbf{k}_{X^d})$ with I injective. Since i_* is left exact it induces a monomorphism $i_*i^{-1}F \to i_*I$ in $\mathrm{Mod}(\mathbf{k}_X)$. Composing with (1.3) and using Lemma 1.22 we have a monomorphism $F \to i_*I$. By Lemma 1.21 the sheaf i_*I is injective and we obtain the result. □

We remark that if $\mathbf{k}$ is a field, any sheaf in $\mathrm{Mod}(\mathbf{k}_{X^d})$ is injective and the morphism (1.3) is already a monomorphism from F to an injective object. In this situation the standard way of building an injective resolution of a given F (that is, we start with $I^0 = i_* i^{-1} F$ and apply the procedure to $\operatorname{coker} \varepsilon_F$, defining $I^1 = i_* i^{-1}(\operatorname{coker} \varepsilon_F)$, then to $\operatorname{coker} d^1, \dots$) gives the so called *Godement resolution* of F.

1.6. **Derived functors.** By Proposition 1.23 all left exact functors from $\mathrm{Mod}(\mathbf{k}_X)$ to an abelian category have a right derived functor (see Definition 2.18 below). In particular we can consider RHom (the derived functor of Hom from $\mathrm{Mod}(\mathbf{k}_X)$ to the category of Abelian groups), $\mathrm{R}\mathcal{H}om$, $\mathrm{R}f_*$ and $\mathrm{R}f_!$. For an open subset $U \subset X$ we have the left exact functors $\Gamma(U; \cdot)$ and $\Gamma_c(U; \cdot)$. Their derived functors are denoted $\mathrm{R}\Gamma(U; \cdot)$ and $\mathrm{R}\Gamma_c(U; \cdot)$. We also use

$$H^i(U; F) := H^i \mathrm{R}\Gamma(U; F), \qquad H^i_c(U; F) := H^i \mathrm{R}\Gamma_c(U; F).$$

We can also prove that the tensor product has a left derived functor, denoted $\overset{\mathrm{L}}{\otimes}$.

An example: the cohomology of an interval. A sheaf F on X is *flabby* if, for any open subset $U \subset X$, the restriction morphism $F(X) \to F(U)$ is surjective. We can check that, when $\mathbf{k}$ is a field, flabby is the same thing as injective. Let $f \colon X \to Y$ be a continuous map. The family of flabby sheaves is f_*-injective, which implies that we can compute $\mathrm{R}f_*(F)$ using a flabby resolution of F. We apply this result to the computation of $H^i(\mathbb{R}; \mathbf{k}_{[a,b]})$ for a closed interval $[a, b]$ of $\mathbb{R}$.

We recall the monomorphism (1.3) $\epsilon \colon \mathbf{k}_{[a,b]} \to i_* i^{-1} \mathbf{k}_{[a,b]}$, where i is the map from $\mathbb{R}$ with the discrete topology to $\mathbb{R}$. We can identify $i_* i^{-1} \mathbf{k}_{[a,b]}$ with the sheaf $\mathcal{F}_{[a,b]}$ of functions on $[a, b]$ defined by $\mathcal{F}_{[a,b]}(U) = \{f \colon U \cap [a, b] \to \mathbb{R}\}$. This sheaf is flabby since we can extend a function defined on $U \cap [a, b]$ arbitrarily to a function defined on $[a, b]$. We define $G = \operatorname{coker}(\epsilon)$ and we have the short exact sequence:

$$0 \to \mathbf{k}_{[a,b]} \to \mathcal{F}_{[a,b]} \to G \to 0. \tag{1.4}$$

Lemma 1.24. *For any open subset $U \subset \mathbb{R}$ the sequence* (1.4) *gives the exact sequence of sections:*

$$0 \to \Gamma(U; \mathbf{k}_{[a,b]}) \xrightarrow{a(U)} \Gamma(U; \mathcal{F}_{[a,b]}) \xrightarrow{b(U)} \Gamma(U; G) \to 0. \tag{1.5}$$

Proof. Long exercise. □

Lemma 1.25. *The sheaf G of* (1.4) *is flabby.*

Proof. Let $U \subset \mathbb{R}$ and $s \in G(U)$ be given. By Lemma 1.24 there exists $s' \in \mathcal{F}_{[a,b]}(U)$ such that $b(U)(s') = s$. Since $\mathcal{F}_{[a,b]}$ is flabby, there exists $t' \in \mathcal{F}_{[a,b]}(\mathbb{R})$ such that $t'|_U = s'$. Then $t = b(\mathbb{R})(t')$ satisfies $t|_U = s$. □

Hence (1.4) gives a flabby resolution of $\mathbf{k}_{[a,b]}$. We deduce that for any open subset U of $\mathbb{R}$

$$H^i(U; \mathbf{k}_{[a,b]}) \simeq H^i(0 \to \Gamma(U; \mathcal{F}_{[a,b]}) \xrightarrow{b(U)} \Gamma(U; G) \to 0).$$

By Lemma 1.24 the morphism $b(U)$ is surjective and we obtain that the cohomology of $\mathbf{k}_{[a,b]}$ is concentrated in degree 0:

Proposition 1.26. *Let $[a, b]$ be a closed interval in $\mathbb{R}$. For any open interval U of $\mathbb{R}$ such that $U \cap [a, b] \neq \emptyset$, we have*

$$H^0(U; \mathbf{k}_{[a,b]}) \simeq \mathbf{k} \qquad \text{and} \qquad H^i(U; \mathbf{k}_{[a,b]}) \simeq 0 \quad \text{for } i \neq 0.$$

We can prove in the same way that, if B is a closed ball in $\mathbb{R}^n$, then $H^*(\mathbb{R}^n; \mathbf{k}_B)$ is concentrated in degree 0, where it is $\mathbf{k}$. We can deduce that $H^*(X; \mathbf{k}_X)$ is concentrated in degree 0, as soon as X is contractible (see [3, §2.7]). Using the sequences of the next paragraph it follows that the Eilenberg–Steenrod axioms are satisfied and we have $H^*(X; \mathbf{k}_X) \simeq H^*(X; \mathbf{k})$ for any CW complex X. Let us rewrite this as follows.

Theorem 1.27. *Let $Z \subset X$ be a closed subset. If Z is a CW complex, then $H^*(X; \mathbf{k}_Z)$ is isomorphic to the singular cohomology $H^*(Z; \mathbf{k})$ of Z.*

1.7. **Relations between functors.** Let us introduce some notations.

Definition 1.28. For a locally closed subset $Z \subset X$ and $F \in \mathrm{Mod}(\mathbf{k}_X)$ we set $\Gamma_Z(F) = \mathcal{H}om(\mathbf{k}_Z, F)$. For an open subset $U \subset X$ we set $\Gamma_Z(U; F) = \Gamma(U; \Gamma_Z(F))$.

Lemma 1.29. *Let Z be locally closed and U be open.*
If Z is closed, we have $\Gamma(U; \Gamma_Z(F)) \simeq \{s \in F(U); \mathrm{supp}(s) \subset Z \cap U\}$.
If Z is open, we have $\Gamma(U; \Gamma_Z(F)) \simeq F(U \cap Z)$.

The functor $\Gamma_Z(\cdot)$ is left exact and its derived functor is $\mathrm{R}\Gamma_Z(F) = \mathrm{R}\mathcal{H}om(\mathbf{k}_Z, F)$. For an open subset U the functor $\Gamma_Z(U; \cdot)$ is also left exact and we have $\mathrm{R}\Gamma_Z(U; F) \simeq \mathrm{R}\Gamma(U; \mathrm{R}\Gamma_Z(F))$. We set

$$H^i_Z(U; F) = H^i \mathrm{R}\Gamma_Z(U; F).$$

Let $U \subset X$ be open and let $F \in \mathsf{D}(\mathbf{k}_X)$. We can deduce from Lemma 1.15 the following long exact sequences (we use the notations of the Lemma):

$$\dots \to H^i(U; F_{W\setminus W'}) \to H^i(U; F_W) \to H^i(U; F_{W'}) \\ \to H^{i+1}(U; F_{W\setminus W'}) \to \dots,$$

$$\dots \to H^i_{W'}(U; F) \to H^i_W(U; F) \to H^i_{W\setminus W'}(U; F) \\ \to H^{i+1}_{W'}(U; F) \to \dots .$$

We can also deduce from Lemma 1.16 the sequences

$$\dots \to H^i_{Z_1\cap Z_2}(U; F) \to H^i_{Z_1}(U; F) \oplus H^i_{Z_2}(U; F) \\ \to H^i_{Z_1\cup Z_2}(U; F) \to H^{i+1}_{Z_1\cap Z_2}(U; F) \to \dots,$$

$$\dots \to H^i(U_1 \cup U_2; F) \to H^i(U_1; F) \oplus H^i(U_2; F) \\ \to H^i(U_1 \cap U_2; F) \to H^{i+1}(U_1 \cup U_2; F) \to \dots .$$

Using these sequences we can deduce from Theorem 1.27

Lemma 1.30. *Let $U \subset X$ be an open subset such that $\overline{U}$ is compact. Then $H^*(X; \mathbf{k}_U) \simeq H^*_c(U; \mathbf{k})$.*

We denote by ω_X the dualizing complex on X. If X is a manifold, ω_X is actually the orientation sheaf shifted by the dimension, that is, $\omega_X \simeq or_X[d_X]$. The duality functors are defined by

$$\mathrm{D}_X(\,\bullet\,) = \mathrm{R}\mathcal{H}om(\,\bullet\,, \omega_X), \qquad \mathrm{D}'_X(\,\bullet\,) = \mathrm{R}\mathcal{H}om(\,\bullet\,, \mathbf{k}_X). \tag{1.6}$$

An important result is the existence of a right adjoint for the derived proper direct image $\mathrm{R}f_!$ (Poincaré-Verdier duality). It is defined under fairly general hypothesis. At least, if $f\colon X \to Y$ is a map of manifolds, there exists $f^!\colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_Y) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$ right adjoint to $\mathrm{R}f_!$, which implies in particular

$$\mathrm{Hom}(\mathrm{R}f_!F, G) \simeq \mathrm{Hom}(F, f^!G)$$

for all $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$, $G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_Y)$. When f is a locally closed embedding we have

$$f^!G \simeq f^{-1}(\mathrm{R}\Gamma_X(G)).$$

When f is a submersion, we have, setting $\omega_{X|Y} = \mathrm{R}\mathcal{H}om(f^{-1}(\omega_Y), \omega_X)$

$$f^!G \simeq f^{-1}(G) \otimes \omega_{X|Y}.$$

In particular if f is a submersion with oriented fiber of dimension d, $f^!G \simeq f^{-1}(G)[d]$.

We recall some useful facts (see [3, §2, §3]).

Proposition 1.31. *Let $f\colon X \to Y$ be a morphism of manifolds, F, G, $H \in \mathsf{D}(\mathbf{k}_X)$, $F', G' \in \mathsf{D}(\mathbf{k}_Y)$. Then we have*

(a) $\mathrm{RHom}(\mathbf{k}_U, F) \simeq \mathrm{R}\Gamma(U; F)$, *for $U \subset X$ open,*

(b) $\mathrm{R}\Gamma(U;\mathrm{R}\mathcal{H}om(F,G)) \simeq \mathrm{RHom}(F|_U, G|_U)$, *for* $U \subset X$ *open,*
(c) H^iF *is the sheaf associated with* $V \mapsto H^i(V;F)$*,*
(d) $H^i\mathrm{RHom}(F,G)$ *is the sheaf associated with* $V \mapsto \mathrm{Hom}(F|_V, G|_V[i])$*,*
(e) $\mathrm{R}\mathcal{H}om(F \overset{\mathrm{L}}{\otimes} G, H) \simeq \mathrm{R}\mathcal{H}om(F, \mathrm{R}\mathcal{H}om(G,H))$*,*
(f) $\mathrm{R}f_!(F \overset{\mathrm{L}}{\otimes} f^{-1}F') \simeq (\mathrm{R}f_!F) \overset{\mathrm{L}}{\otimes} F'$*, (projection formula),*
(g) $f^!\,\mathrm{R}\mathcal{H}om(F',G') \simeq \mathrm{R}\mathcal{H}om(f^{-1}F', f^!G')$*,*
(h) $\mathrm{R}f_*\mathrm{R}\mathcal{H}om(F,G) \simeq \mathrm{R}\mathcal{H}om(\mathrm{R}f_!F, \mathrm{R}f_*G)$*, if* f *is an embedding,*
(i) *for a Cartesian diagram* $\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ X' & \xrightarrow{f'} & Y' \end{array}$ *we have the base change formula* $f'^{-1}\mathrm{R}g'_!(F') \simeq \mathrm{R}g_!f^{-1}(F')$*,*

The adjunction between $\overset{\mathrm{L}}{\otimes}$ and $\mathrm{R}\mathcal{H}om$ together with $\mathbf{k}_U \otimes \mathbf{k}_{\overline{U}} \simeq \mathbf{k}_U$ give

$$\mathrm{Hom}(\mathbf{k}_U, \mathrm{D}'(\mathbf{k}_{\overline{U}})) \simeq \mathrm{Hom}(\mathbf{k}_U, \mathbf{k}_X) \simeq H^0(U;\mathbf{k}_X)$$

and the canonical section $1 \in H^0(U;\mathbf{k}_X)$ gives a morphism $\mathbf{k}_U \to \mathrm{D}'(\mathbf{k}_{\overline{U}})$. Similarly we have a natural morphism $\mathbf{k}_{\overline{U}} \to \mathrm{D}'(\mathbf{k}_U)$. In the following case they are isomorphisms.

Lemma 1.32. *If the inclusion* $U \subset X$ *is locally homeomorphic to the inclusion* $]-\infty, 0[\times\mathbb{R}^{n-1} \subset \mathbb{R}^n$ *(for example, if* ∂U *is smooth), then the above morphisms* $\mathbf{k}_U \to \mathrm{D}'(\mathbf{k}_{\overline{U}})$ *and* $\mathbf{k}_{\overline{U}} \to \mathrm{D}'(\mathbf{k}_U)$ *are isomorphisms:*

$$\mathbf{k}_{\overline{U}} \xrightarrow{\sim} \mathrm{D}'(\mathbf{k}_U), \qquad \mathbf{k}_U \xrightarrow{\sim} \mathrm{D}'(\mathbf{k}_{\overline{U}}). \tag{1.7}$$

Proof. Let us prove the first isomorphism. It is enough to check that $\mathbf{k}_{\overline{U}} \to \mathrm{D}'(\mathbf{k}_U)$ induces an isomorphism $\mathbf{k} \xrightarrow{\sim} (\mathrm{D}'(\mathbf{k}_U))_x$ for each $x \in X$. Since $\mathrm{D}'(\mathbf{k}_U) = \mathrm{R}\mathcal{H}om(\mathbf{k}_U, \mathbf{k}_X)$, Proposition 1.31-(b-c) gives

$$H^i(\mathrm{D}'(\mathbf{k}_U))_x \simeq \varinjlim_{x\in V} \mathrm{Hom}(\mathbf{k}_U|_V, \mathbf{k}_X|_V[i]).$$

By (a) we have $\mathrm{Hom}(\mathbf{k}_U|_V, \mathbf{k}_X|_V[i]) \simeq H^i(U\cap V;\mathbf{k}_X)$. By Theorem 1.27 this is the cohomology of $U \cap V$ which can be chosen contractible in our inductive limit. ☐

Example 1.33. We have $\mathrm{R}\Gamma_{\{0\}}(\mathbf{k}_{\mathbb{R}^n}) \simeq \mathbf{k}_{\{0\}}[-n]$. Indeed the sheaf $\mathrm{R}\Gamma_{\{0\}}\mathbf{k}_{\mathbb{R}^n}$ has support $\{0\}$ and its stalk at 0 coincides with its global sections. We have the excision exact sequence

$$H^i_{\{0\}}(\mathbb{R}^n;\mathbf{k}_{\mathbb{R}^n}) \to H^i(\mathbb{R}^n;\mathbf{k}_{\mathbb{R}^n}) \to H^i_{\mathbb{R}^n\setminus\{0\}}(\mathbb{R}^n;\mathbf{k}_{\mathbb{R}^n}).$$

By Proposition 1.31-(a) $H^i_{\mathbb{R}^n\setminus\{0\}}(\mathbb{R}^n;\mathbf{k}_{\mathbb{R}^n}) \simeq H^i(\mathbb{R}^n \setminus \{0\};\mathbf{k}_{\mathbb{R}^n})$ and this is the cohomology of the sphere. The result follows.

Example 1.34. The previous example generalizes as follows. Let X be a manifold and Z a submanifold of codimension d. Then $\mathrm{R}\Gamma_Z(\mathbf{k}_X) \simeq or_{Z|X}[-d]$ where $or_{Z|X}$ is the relative orientation sheaf.

Example 1.35. In $\mathbb{R}^2$ we define $Z = \{x \geq 0;\, y \geq 0\}$ and $U = \{x < 0;\, y < 0\}$. Then $\mathrm{RHom}(\mathbf{k}_Z, \mathbf{k}_U) \simeq \mathbf{k}[-2]$. Indeed, by Lemma 1.32 we have

$$\begin{aligned}\mathrm{RHom}(\mathbf{k}_Z, \mathbf{k}_U) &\simeq \mathrm{RHom}(\mathbf{k}_Z, \mathrm{R}\mathcal{H}om(\mathbf{k}_{\overline{U}}, \mathbf{k}_{\mathbb{R}^2})) \\ &\simeq \mathrm{RHom}(\mathbf{k}_Z \otimes \mathbf{k}_{\overline{U}}, \mathbf{k}_{\mathbb{R}^2}) \\ &\simeq \mathrm{RHom}(\mathbf{k}_{Z\cap\overline{U}}, \mathbf{k}_{\mathbb{R}^2}) \\ &\simeq \mathrm{RHom}(\mathbf{k}_{\{0\}}, \mathbf{k}_{\mathbb{R}^2})\end{aligned}$$

and the result follows from Example 1.33.

Example 1.36. By the previous example $\mathrm{Hom}(\mathbf{k}_Z, \mathbf{k}_U[2]) \simeq \mathbf{k}$. Let $u\colon \mathbf{k}_Z \to \mathbf{k}_U[2]$ be the image of $1 \in \mathbf{k}$. Let $F \in \mathsf{D}(\mathbf{k}_{\mathbb{R}^2})$ be given by the dt $F \to \mathbf{k}_Z \to \mathbf{k}_U[2] \xrightarrow{+1}$. Then F is isomorphic to the complex $\mathbf{k}_{\mathbb{R}^2} \xrightarrow{d} \mathbf{k}_{Z_1} \oplus \mathbf{k}_{Z_2}$ where $\mathbf{k}_{\mathbb{R}^2}$ is in degree 0, $Z_1 = \{x \geq 0\}$, $Z_2 = \{y \geq 0\}$ and d is the sum of the natural morphisms $\mathbf{k}_{\mathbb{R}^2} \to \mathbf{k}_{Z_i}$ induced by the inclusions of closed subsets $Z_i \subset \mathbb{R}^2$.

2. Derived categories

2.1. Categories of complexes.

Definition 2.1. Let $\mathcal{C}$ be an additive category. A complex $(X^\cdot, d^\cdot_X)$ in $\mathcal{C}$ is a sequence of composable morphisms in $\mathcal{C}$

$$\cdots \to X^i \xrightarrow{d^i_X} X^{i+1} \to \cdots$$

such that $d^{i+1} \circ d^i = 0$, for all $i \in \mathbb{Z}$ (we forget the subscripts when there is no ambiguity). The sequence of morphisms d^i_X is called the differential.

A morphism f from a complex $(X^\cdot, d^\cdot_X)$ to a complex $(Y^\cdot, d^\cdot_Y)$ is a sequence of morphisms $f^i\colon X^i \to Y^i$, $i \in \mathbb{Z}$, commuting with the differentials.

We denote by $\mathsf{C}(\mathcal{C})$ the category of complexes in $\mathcal{C}$. A complex is said bounded from below (resp. above) if $X^i \simeq 0$ for $i \ll 0$ (resp. $i \gg 0$). It is bounded if it bounded from below and from above. We let $\mathsf{C}^+(\mathcal{C})$, $\mathsf{C}^-(\mathcal{C})$, $\mathsf{C}^b(\mathcal{C})$ be the corresponding categories.

Definition 2.2. Let $\mathcal{C}$ be an abelian category and let $X = (X^\cdot, d_X^\cdot) \in \mathsf{C}(\mathcal{C})$. For $i \in \mathbb{Z}$ we define

$$Z^i(X) = \ker d_X^i, \qquad B^i(X) = \operatorname{im} d_X^{i-1},$$
$$H^i(X) = Z^i(X)/B^i(X) = \operatorname{coker}(B^i(X) \to Z^i(X))$$

and we call $H^i(X)$ the i^{th} cohomology of X. In the case of the category of groups $Z^i(X)$ (resp. $B^i(X)$) is called the i^{th} group of cocycles (resp. boundaries).

A morphism of complexes $f\colon X \to Y$ induces morphisms $Z^i(f)$, $B^i(f)$, $H^i(f)$ and Z^i, B^i, H^i are functors from $\mathsf{C}(\mathcal{C})$ to $\mathcal{C}$. We say that f is a quasi-isomorphism if the morphisms $H^i(f)\colon H^i(X) \to H^i(Y)$ are isomorphisms, for all $i \in \mathbb{Z}$.

If $\mathcal{C}$ is abelian, then $\mathsf{C}(\mathcal{C})$ is also abelian. Moreover for a morphism $f\colon X \to Y$ in $\mathsf{C}(\mathcal{C})$ we have $(\ker f)^i = \ker(f^i)$ and $(\operatorname{coker} f)^i = \operatorname{coker}(f^i)$.

Proposition 2.3. *Let $\mathcal{C}$ be an abelian category and let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence in $\mathsf{C}(\mathcal{C})$. Then there exists a canonical long exact sequence in $\mathcal{C}$*

$$\cdots \to H^n(X) \xrightarrow{H^n(f)} H^n(Y) \xrightarrow{H^n(g)} H^n(Z) \xrightarrow{\delta^n} H^{n+1}(X)$$
$$\xrightarrow{H^{n+1}(f)} H^{n+1}(Y) \xrightarrow{H^{n+1}(g)} H^{n+1}(Z) \to \cdots .$$

Definition 2.4. Let $\mathcal{C}$ be an abelian category and let $I \in \operatorname{Ob}(\mathcal{C})$. We say that I is *injective* if the functor $\operatorname{Hom}(\cdot, I)$ is exact, that is, if for any short exact sequence $0 \to A \to B$, the sequence $\operatorname{Hom}(A, I) \to \operatorname{Hom}(B, I) \to 0$ is exact. We say that $\mathcal{C}$ has *enough injectives* if for any $M \in \operatorname{Ob}(\mathcal{C})$, there exist an injective object I and an exact sequence $0 \to M \to I$.

Proposition 2.5. *Let $\mathcal{C}$ be an abelian category. We assume that $\mathcal{C}$ has enough projectives. Then any $X \in \mathsf{C}^+(\mathcal{C})$ has an injective (right) resolution, that is, a morphism $u\colon X \to I$ in $\mathsf{C}^+(\mathcal{C})$ such that u is a quasi-isomorphism and I^k is injective for each $k \in \mathbb{Z}$.*

This proposition holds in $\mathsf{C}(\mathcal{C})$ but the right notion of injective resolution is more complicated. The next proposition says that a projective resolution is unique up to homotopy in the following sense.

Definition 2.6. Let $\mathcal{C}$ be an additive category and let $P = (P^\cdot, d_P^\cdot)$, $Q = (Q^\cdot, d_Q^\cdot) \in \mathsf{C}(\mathcal{C})$. We say that two morphisms $f, g\colon P \to Q$ in $\mathsf{C}(\mathcal{C})$

are homotopic if there exists a family of morphisms $s^i\colon P^i \to Q^{i-1}$, $i \in \mathbb{Z}$, such that

$$f^n - g^n = d_Q^{n-1} \circ s^n + s^{n+1} \circ d_P^n,$$

for all $n \in \mathbb{Z}$.

The homotopy relation is compatible with the additive structure of $\mathrm{Hom}(P,Q)$ and with the composition in $\mathsf{C}(\mathcal{C})$. It follows that we can define a category of *complexes up to homotopy* as follows.

Definition 2.7. Let $\mathcal{C}$ be an additive category. We define a category $\mathsf{K}(\mathcal{C})$ by $\mathrm{Ob}(\mathsf{K}(\mathcal{C})) = \mathrm{Ob}(\mathsf{C}(\mathcal{C}))$ and

$$\mathrm{Hom}_{\mathsf{K}(\mathcal{C})}(P,Q) = \mathrm{Hom}_{\mathsf{C}(\mathcal{C})}(P,Q)/\sim_h,$$

where $\sim_h$ is the homotopy relation on $\mathrm{Hom}_{\mathsf{C}(\mathcal{C})}(P,Q)$. We have an obvious functor $\mathsf{K}(\mathcal{C}) \to \mathsf{C}(\mathcal{C})$ which is the identity on objects and the quotient map on the morphisms.

The category $\mathsf{K}(\mathcal{C})$ is additive. It is no longer abelian but it has a triangulated structure.

Proposition 2.8. *Let $\mathcal{C}$ be an abelian category, let $X, Y \in \mathsf{C}^+(\mathcal{C})$ and let $v\colon Y \to J$ be an injective resolution in $\mathsf{C}^+(\mathcal{C})$. Let $f\colon X \to Y$ be a morphism and $u\colon X \to I$ a quasi-isomorphism. Then there exists a morphism $f'\colon I \to J$ such that $v \circ f = f' \circ u$. Moreover, if $f''\colon I \to J$ is another such morphism, then f' and f'' are homotopic. In particular two injective resolutions of X are canonically isomorphic in $\mathsf{K}(\mathcal{C})$.*

2.2. **Definition of derived categories.** Here we only give a brief account on the subject and refer to the first chapter of [3] or to [**?**] for details and proofs.

Definition 2.9. Let $\mathcal{C}$ be an abelian category and let $u\colon X \to Y$ be a morphism in $\mathsf{C}(\mathcal{C})$ or in $\mathsf{K}(\mathcal{C})$. We say that u is a quasi-isomorphism (qis for short) if the morphisms $H^i(u)\colon H^i(X) \to H^i(Y)$ are isomorphisms, for all $i \in \mathbb{Z}$.

The derived category of $\mathcal{C}$, denoted $\mathsf{D}(\mathcal{C})$, is obtained from $\mathsf{C}(\mathcal{C})$ by inverting the qis. This process is called *localization*.

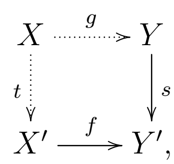
Definition 2.10. Let $\mathcal{A}$ be a category and $\mathcal{S}$ a family of morphisms in $\mathcal{A}$. A localization of $\mathcal{A}$ by $\mathcal{S}$ is a category $\mathcal{A}_{\mathcal{S}}$ (a priori in a bigger universe) and a functor $Q\colon \mathcal{A} \to \mathcal{A}_{\mathcal{S}}$ such that

(i) for all $s \in \mathcal{S}$, $Q(s)$ is an isomorphism,
(ii) for any category $\mathcal{B}$ and any functor $F\colon \mathcal{A} \to \mathcal{B}$ such that $F(s)$ is an isomorphism for all $s \in \mathcal{S}$, there exists a (unique) functor $F_{\mathcal{S}}\colon \mathcal{A}_{\mathcal{S}} \to \mathcal{B}$ such that $F \simeq F_{\mathcal{S}} \circ Q$,

It is possible to construct $\mathcal{A}_{\mathcal{S}}$ as a category with the same objects as $\mathcal{A}$ and with morphisms defined as chains $(s_1, u_1, s_2, u_2, \dots, s_n, u_n)$ with $s_i \in \mathcal{S}$ and u_i any morphism in $\mathcal{A}$ modulo some equivalence relation. Such a chain is meant to represent $u_n \circ s_n^{-1} \circ u_{n-1} \circ \cdots \circ s_1^{-1}$. However we will only consider a special case where the localization is obtained by a calculus of fractions.

Definition 2.11. A family $\mathcal{S}$ of morphisms in $\mathcal{A}$ is a left multiplicative system if

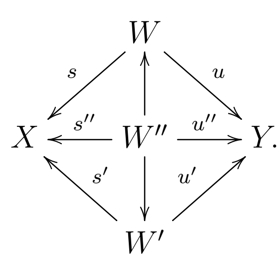
(i) any isomorphism belongs to $\mathcal{S}$,
(ii) if $f, g \in \mathcal{S}$ and $g \circ f$ is defined, then $g \circ f \in \mathcal{S}$,
(iii) for given morphisms f, s, $s \in \mathcal{S}$, as in the following diagram, there exist g, t, $t \in \mathcal{S}$, making the diagram commutative



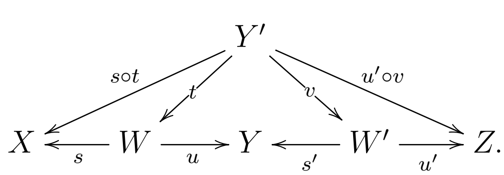
(iv) for two given morphisms $f, g\colon X \to Y$ in $\mathcal{A}$, if there exists $s \in \mathcal{S}$ such that $s \circ f = s \circ g$, then there exists $t \in \mathcal{S}$ such that $f \circ t = g \circ t$:

$$W \overset{t}{\dashrightarrow} X \xrightarrow{f,g} Y \xrightarrow{s} Z.$$

Proposition 2.12. *Let $\mathcal{A}$ be a category and $\mathcal{S}$ a left multiplicative system. Then $\mathcal{A}_{\mathcal{S}}$ can be described as follows. The set of objects is $\mathrm{Ob}(\mathcal{A}_{\mathcal{S}}) = \mathrm{Ob}(\mathcal{A})$. For $X, Y \in \mathrm{Ob}(\mathcal{A})$, we have $\mathrm{Hom}_{\mathcal{A}_{\mathcal{S}}}(X, Y) = \{(W, s, u);\ s\colon W \to X$ is in $\mathcal{S}$ and $u\colon W \to Y$ is in $\mathcal{A}\}/\sim$, where the equivalence relation $\sim$ is given by $(W, s, u) \sim (W', s', u')$ if there exists (W'', s'', u''), $s'' \in \mathcal{S}$, such that we have a commutative diagram*



The composition "$u's'^{-1}us^{-1}$" is visualized by the diagram



where t, v, $t \in \mathcal{S}$, are given by (iii) in Definition 2.11.

Let us go back to our abelian category $\mathcal{C}$.

Proposition 2.13. *Let Qis be the family of qis in $\mathsf{K}(\mathcal{C})$. Then Qis is a left (and right) multiplicative system.*

Definition 2.14. Let $\mathcal{C}$ be an abelian category. The derived category of $\mathcal{C}$ is the localization $\mathsf{D}(\mathcal{C}) = (\mathsf{K}(\mathcal{C}))_{Qis}$. We denote by $Q_{\mathcal{C}} \colon \mathsf{K}(\mathcal{C}) \to \mathsf{D}(\mathcal{C})$ the localization functor (or its composition with $\mathsf{C}(\mathcal{C}) \to \mathsf{K}(\mathcal{C})$). Starting with $\mathsf{K}^*(\mathcal{C})$ where $* = +, -$ or b, we define in the same way $\mathsf{D}^*(\mathcal{C})$.

The categories $\mathsf{K}(\mathcal{C})$ and $\mathsf{D}(\mathcal{C})$ are additive. They are not abelian in general.

By definition the cohomology functors $H^i \colon \mathsf{K}(\mathcal{C}) \to \mathcal{C}$, $i \in \mathbb{Z}$, factorize through the localization functor. We still denote by $H^i \colon \mathsf{D}(\mathcal{C}) \to \mathcal{C}$ the induced functors.

Lemma 2.15. *Let $\mathcal{C}, \mathcal{C}'$ be abelian categories. Let $F \colon \mathcal{C} \to \mathcal{C}'$ be an exact functor. Then $C(F)$ sends qis to qis. In particular $Q_{\mathcal{C}'} \circ \mathsf{K}(F) \colon \mathsf{K}(\mathcal{C}) \to \mathsf{D}(\mathcal{C}')$ sends qis to isomorphisms and factorizes in a unique way through a functor $\mathsf{D}(\mathcal{C}) \to \mathsf{D}(\mathcal{C}')$ that we still denote by F:*

$$\begin{array}{ccc} \mathsf{K}(\mathcal{C}) & \xrightarrow{\mathsf{K}(F)} & \mathsf{K}(\mathcal{C}') \\ {\scriptstyle Q_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle Q_{\mathcal{C}'}} \\ \mathsf{D}(\mathcal{C}) & \xrightarrow{F} & \mathsf{D}(\mathcal{C}'). \end{array}$$

Remark 2.16. We have a natural embedding of $\mathcal{C}$ in $\mathsf{C}(\mathcal{C})$ which sends $X \in \mathcal{C}$ to the complex $(X^{\cdot}, d_X^{\cdot})$ with $X^0 = X$ and $X^i = 0$ for $i \neq 0$. This induces by composition other functors $\mathcal{C} \to \mathsf{K}(\mathcal{C})$ and $\mathcal{C} \to \mathsf{D}(\mathcal{C})$. We can check that all these functors are fully faithful embeddings of $\mathcal{C}$ in $\mathsf{C}(\mathcal{C})$, $\mathsf{K}(\mathcal{C})$ or $\mathsf{D}(\mathcal{C})$.

Proposition 2.8 translate as follows.

Proposition 2.17. *Let $\mathcal{C}$ be an abelian category. We assume that $\mathcal{C}$ has enough injectives and we let $\mathcal{I}$ be its full subcategory of injective objects. We denote by $Q|_{\mathcal{I}} \colon \mathsf{K}^+(\mathcal{I}) \to \mathsf{D}^+(\mathcal{C})$ the functor induced by the quotient functor. Then $Q|_{\mathcal{I}}$ is an equivalence of categories.*

Definition 2.18 (Derived functor)**.** Let $\mathcal{C}, \mathcal{C}'$ be abelian categories. We assume that $\mathcal{C}$ has enough injectives. Let $F \colon \mathcal{C} \to \mathcal{C}'$ (or $F \colon \mathsf{C}^+(\mathcal{C}) \to \mathsf{C}^+(\mathcal{C}')$) be a left exact functor. Let $\mathsf{K}(F) \colon \mathsf{K}^+(\mathcal{I}) \to \mathsf{K}^+(\mathcal{C}')$ be the functor induced by F. We define $RF \colon \mathsf{D}^+(\mathcal{C}) \to \mathsf{D}^+(\mathcal{C}')$ by $RF =$

$Q_{\mathcal{C}'} \circ \mathsf{K}(F) \circ \mathbf{res}$, where **res** is an inverse to the equivalence $Q|_{\mathcal{I}}$ of Proposition 2.17.

If F is exact then $RF \simeq F$ (with the notation of Lemma 2.15). For a left exact functor $F\colon \mathcal{C} \to \mathcal{C}'$ and $X \in \mathcal{C}$ we have $H^0RF(X) \simeq F(X)$ (using the embedding of Remark 2.16).

Truncation functors. Let $\mathcal{C}$ be an abelian category. For a given $n \in \mathbb{Z}$ we define $\tau_{\leq n}, \tau_{\geq n}\colon \mathsf{C}(\mathcal{C}) \to \mathsf{C}(\mathcal{C})$ by

$$\tau_{\leq n}(X) = \cdots \to X^{n-2} \to X^{n-1} \to \ker(d_X^n) \to 0 \to \cdots$$
$$\tau_{\geq n}(X) = \cdots \to 0 \to \operatorname{coker}(d_X^{n-1}) \to X^{n+1} \to X^{n+2} \to \cdots .$$

We have natural morphisms in $\mathsf{C}(\mathcal{C})$, for $n \leq m$,

$$\tau_{\leq n}(X) \to X, \qquad X \to \tau_{\geq n}(X),$$
$$\tau_{\leq n}(X) \to \tau_{\leq m}(X), \qquad \tau_{\geq n}(X) \to \tau_{\geq m}(X).$$

We have $H^i(\tau_{\leq n}(X)) \simeq H^i(X)$ for $i \leq n$ and $H^i(\tau_{\leq n}(X)) \simeq 0$ for $i > 0$. We have a similar result for $\tau_{\geq n}(X)$ and the above morphisms induce the tautological morphisms on the cohomology (that is, the identity morphism of H^i if both groups are non-zero, or the zero morphism).

In particular the functors $\tau_{\leq n}$, $\tau_{\geq n}$ send qis to qis and they induce functors, denoted in the same way, on $\mathsf{D}(\mathcal{C})$, together with the same morphisms of functors. We see from the definition, for any $X \in \mathsf{D}(\mathcal{C})$ and any $i \in \mathbb{Z}$:

$$\tau_{\leq i}\tau_{\geq i}(X) \simeq \tau_{\geq i}\tau_{\leq i}(X) \simeq H^i(X)[-i]. \tag{2.1}$$

Lemma 2.19. *Let $\mathcal{C}$ be an abelian category and let $X \in \mathsf{D}(\mathcal{C})$ be an objet concentrated in one degree i_0, that is, $H^i(X) \simeq 0$ if $i \neq i_0$. Then $X \simeq H^{i_0}(X)[-i_0]$.*

Proof. By the hypothesis and by the description of the cohomology of $\tau_{\leq n}(X)$, $\tau_{\geq n}(X)$, the morphisms $\tau_{\leq i_0}(X) \to X$ and $\tau_{\leq i_0}(X) \to \tau_{\geq i_0}(\tau_{\leq i_0}(X))$ are isomorphisms in $\mathsf{D}(\mathcal{C})$. Hence $X \simeq \tau_{\geq i_0}(\tau_{\leq i_0}(X))$ and we conclude with (2.1). □

2.3. **Triangulated structure.** We recall that a triangulated category $\mathcal{T}$ is an additive category endowed with an auto-equivalence $X \mapsto X[1]$ and a family of distinguish triangles (dt) $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ such that

(TR1) every morphism can be extended to distinguished triangle, the collection of distinguished triangles is stable under isomorphism and, for any $X \in \mathcal{T}$ the triangle $X \xrightarrow{\mathrm{id}} X \xrightarrow{0} 0 \xrightarrow{0} X[1]$ is distinguished, (TR2) $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a dt if and only if $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ is a dt,

(TR3) for two dt $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} X'[1]$, any commutative square $f' \circ u = v \circ f$ (with $u\colon X \to X'$, $v\colon Y \to Y'$) can be extended to a morphism of triangles (that is, there exists $w\colon Z \to Z'$ making two other commutative squares),

(TR4) octahedral axiom (it is the distinguished triangle version of the isomorphism $(C/A)/(B/A) \simeq C/B$ for two inclusions of $\mathbf{k}$-modules $A \hookrightarrow B \hookrightarrow C$).

If $\mathcal{C}$ is an abelian category, then $\mathsf{D}(\mathcal{C})$ is triangulated. If $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is a short exact sequence in $\mathsf{C}(\mathcal{C})$, then there exists a morphism $Z \xrightarrow{h} X[1]$ in $\mathsf{D}(\mathcal{C})$ such that $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a dt. If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a dt in $\mathsf{D}(\mathcal{C})$, then we have a long exact sequence in $\mathcal{C}$:

$$\cdots \to H^n(X) \xrightarrow{H^n(f)} H^n(Y) \xrightarrow{H^n(g)} H^n(Z) \xrightarrow{H^n(h)} H^{n+1}(X) \xrightarrow{H^{n+1}(f)} H^{n+1}(Y) \xrightarrow{H^{n+1}(g)} H^{n+1}(Z) \to \cdots .$$

The derived functor RF of Definition 2.18 is triangulated (i.e. it sends a dt to a dt).

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